Example Sheet 1

1. Let $B_1, B_2, \ldots$ be disjoint events with $\bigcup_{n=1}^{\infty} B_n = \Omega$.
Show that if A is another event and $P(A \mid B_n) = p$ for all n then $P(A) = p$.
Deduce that if X and Y are discrete random variables then the following are equivalent:

(a) X and Y are independent,

(b) the conditional distribution of X given $Y = y$ is independent of y.

2. Show that if $(X_n)_{n\geq 0}$ is a discrete-time Markov chain with transition matrix P and $Y_n = X_{kn}$, then $(Y_n)_{n\geq 0}$ is a Markov chain with transition matrix P^k.

3. Let X_0 be a random variable with values in a countable set I. Let $Y_1, Y_2, \ldots$ be a sequence of independent random variables, uniformly distributed on $[0, 1]$. Suppose we are given a function

$$G : I \times [0, 1] \to I$$

and define inductively for $n \geq 0$

$$X_{n+1} = G(X_n, Y_{n+1}).$$

Show that $(X_n)_{n\geq 0}$ is a Markov chain and express its transition matrix P in terms of G. Can all Markov chains be realized in this way? How would you simulate a Markov chain using a computer?

4. Suppose that $Z_0, Z_1, \ldots$ are independent, identically distributed random variables such that $Z_i = 1$ with probability p and $Z_i = 0$ with probability $1 - p$. Set $S_0 = 0$, $S_n = Z_1 + \ldots + Z_n$. In each of the following cases determine whether $(X_n)_{n\geq 0}$ is a Markov chain:

(a) $X_n = Z_n$,

(b) $X_n = S_n$,

(c) $X_n = S_0 + \cdots + S_n$,

(d) $X_n = (S_n, S_0 + \cdots + S_n)$.

In the cases where $(X_n)_{n\geq 0}$ is a Markov chain find its state-space and transition matrix, and in the cases where it is not a Markov chain give an example where $P(X_{n+1} = i \mid X_n = j, X_{n-1} = k)$ is not independent of k.

5. A flea hops randomly on vertices of a triangle, hopping to each of the other vertices with equal probability. Find the probability that after n hops the flea is back where it started.

A second flea also hops about on the vertices of a triangle, but this flea is twice as likely to jump clockwise as anti-clockwise. What is the probability that after n hops this second flea is back where it started? [*Hint*: $\frac{1}{2} \pm \frac{i}{2\sqrt{3}} = \frac{1}{\sqrt{3}}\, e^{\pm i\pi/6}$]

6. A die is 'fixed' so that each time it is rolled the score cannot be the same as the preceding score, all other scores having probability $1/5$. If the first score is 6, what is the probability $p_{66}^{(n+1)}$ that the $(n+1)$st score is 6? What is the probability that the $(n+1)$st score is 1?

7. Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{1, 2, 3\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ p & 1-p & 0 \end{pmatrix}.$$

Calculate $P(X_n = 1 \mid X_0 = 1)$ when (a) $p = 1/16$, (b) $p = 1/6$, (c) $p = 1/12$.

8. Identify the communicating classes of the transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

(a) Which of them are closed?

(b) Which states are recurrent and which are transient?

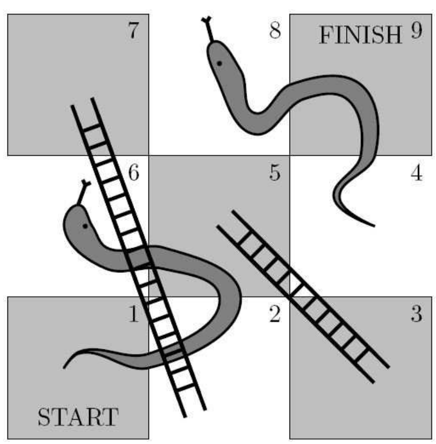
9. Show that every transition matrix on a finite state-space has at least one closed communicating class. Find an example of a transition matrix with no closed communicating class.

10. A gambler has £2 and needs to increase it to £10 in a hurry. He can play a game with the following rules: a fair coin is tossed; if a player bets on the right side, he wins a sum equal to his stake, and his stake is returned; otherwise he loses his stake. The gambler decides to use a bold strategy in which he stakes all his money if he has £5 or less, and otherwise stakes just enough to increase his capital, if he wins, to £10. Let $X_0 = 2$ and let X_n be his capital after n throws. Prove that the gambler will achieve his aim with probability 1/5.

(a) What is the expected number of tosses until the gambler either achieves his aim or loses his capital?

(b) What is the expected number of tosses conditional on the gambler achieving his aim?
[*Hint: Let A be the event that he achieves his aim and find* $P(X_{n+1} = j \mid X_n = i, A)$.]

11. A simple game of 'snakes and ladders' is played on a board of nine squares. At each turn a player tosses a fair coin and advances one or two places in the usual fashion, according to whether the coin lands heads or tails.



How many turns on average does it take to complete the game?

What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

12. Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0, 1, \ldots\}$ with transition probabilities given by

$$p_{01} = 1, \quad p_{ii+1} + p_{ii-1} = 1, \quad p_{ii+1} = \left(\frac{i+1}{i}\right)^2 p_{ii-1}, \quad i \geq 1.$$

Show that if $X_0 = 0$ the probability that $X_n \geq 1$ for all $n \geq 1$ is $6/\pi^2$.

13. This question builds on Question 10. You could use Mathematica or MATLAB to help you do the calculations, but if you are careful you will not need to solve any difficult systems of linear equations.

Smith is in jail and has 3 pounds; he can get out on bail if he has 8 pounds. A guard agrees to make a series of bets with him. If Smith bets A pounds, he wins A pounds with probability p and loses A pounds with probability $q = 1 - p$. Assume $p < q$. Find the probability that he wins 8 pounds before losing all of his money if

(a) he bets 1 pound each time (timid strategy). (answer: $\frac{(q/p)^3-1)}{(q/p)^8-1}$.)

(b) he bets, each time, as much as possible but not more than necessary to bring his fortune up to 8 pounds (bold strategy). (answer: $p^2 + pqp$).

(c) For $p = 2/5$ which of the above strategies gives Smith the better chance of getting out of jail? What about any $p < q$?

(d) Suppose Smith bets according to the bold strategy, except that with 3 pounds he bets only 1. Find the probability that he gets out of jail with this strategy and compare it to (b).

The answer to (d) is surprising. It was contrary to the accepted wisdom and impressed Leonard Savage when it was presented to him by a Ph.D. student named Lester Dubins, in 1955. They developed a collaboration culminating in the famous monograph How to Gamble if You Must (Inequalities for Stochastic Processes). *You can read more about these so-called red and black games at the Virtual Laboratories in Probability and Statistics:* `www.math.uah.edu/stat/games/RedBlack.xhtml`.

Extras

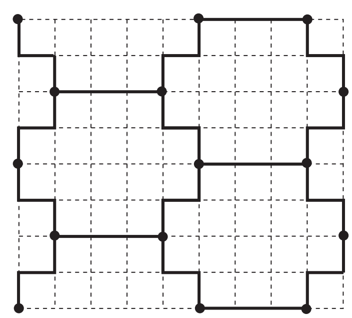
The following questions are optional, but could be fun to do or to discuss with your supervisor. You should at least read them, to appreciate the type of problems that the methods of our course can address.

14. Consider this N-person gamblers' ruin game. Player i starts with n_i (≥ 1) pounds, $i = 1, \ldots, N$. At each step of the game a player is chosen at random and that player wins one pound from each of the others. Prove that the mean time until the first player goes bankrupt is $k(n_1, n_2) = n_1 n_2$ when $N = 2$, and when $N = 3$:

$$k(n_1, n_2, n_3) = \frac{n_1 n_2 n_3}{n_1 + n_2 + n_3 - 2}.$$

What about $N > 3$?

15. Let G be a graph. Suppose $\bar{G}$ is formed from G by deleting some of its edges. It is a theorem that if the random walk in G started from vertex i is recurrent, then the random walk in $\bar{G}$ started from i is also recurrent. Use this theorem to prove that the random walk on the infinite honeycomb graph (composed of hexagons) is recurrent.



A hint is in the diagram, but think carefully how you must argue.

16. Consider a graph G with n vertices, and m edges which have distinct labels $1, 2, \ldots, m$. Place a pedestrian at each vertex. We will now move the pedestrians around the graph in m steps as follows: *at step i ($i = 1, \ldots, m$) let the two pedestrians at the ends of edge i exchange places.* Suppose that pedestrian i moves k_i times.

(a) What is the mean of $k_1, \ldots, k_n$?

(b) Prove this theorem: *No matter how the edges are numbered it is always possible to find a walk along a sequence of increasingly numbered edges whose length is at least the average degree of G.*

(If you do not know what 'walk' or 'degree' means as regards a graph: Google it to find out!)