Paper 3, Section I

9H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain with state space S.

(i) What does it mean to say that $(X_n)_{n\geqslant 0}$ has the strong Markov property? Your answer should include the definition of the term *stopping time.*

(ii) Show that

$$\mathbb{P}(X_n = i \text{ at least } k \text{ times } | X_0 = i) = \left[\mathbb{P}(X_n = i \text{ at least once } | X_0 = i)\right]^k$$

for a state $i \in S$. You may use without proof the fact that $(X_n)_{n\geqslant 0}$ has the strong Markov property.

Paper 4, Section I

9H Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain on a state space S, and let $p_{ij}(n) = \mathbb{P}(X_n = j | X_0 = i)$.

(i) What does the term *communicating class* mean in terms of this chain?

(ii) Show that $p_{ii}(m+n) \geqslant p_{ij}(m)p_{ji}(n)$.

(iii) The period d_i of a state i is defined to be

$$d_i = \gcd\{n \geqslant 1 : p_{ii}(n) > 0\}.$$

Show that if i and j are in the same communicating class and $p_{jj}(r) > 0$, then d_i divides r.

Paper 1, Section II

20H Markov Chains

Let $P = (p_{ij})_{i,j \in S}$ be the transition matrix for an irreducible Markov chain on the finite state space S.

(i) What does it mean to say π is the invariant distribution for the chain?

(ii) What does it mean to say the chain is in detailed balance with respect to π?

(iii) A symmetric random walk on a connected finite graph is the Markov chain whose state space is the set of vertices of the graph and whose transition probabilities are

$$p_{ij} = \begin{cases} 1/D_i & \text{if } j \text{ is adjacent to } i \\ 0 & \text{otherwise,} \end{cases}$$

where D_i is the number of vertices adjacent to vertex i. Show that the random walk is in detailed balance with respect to its invariant distribution.

(iv) Let π be the invariant distribution for the transition matrix P, and define an inner product for vectors $x, y \in \mathbb{R}^S$ by the formula

$$\langle x, y \rangle = \sum_{i \in S} x_i \pi_i y_i .$$

Show that the equation

$$\langle x, Py \rangle = \langle Px, y \rangle$$

holds for all vectors $x, y \in \mathbb{R}^S$ if and only if the chain is in detailed balance with respect to π. [Here $z \in \mathbb{R}^S$ means $z = (z_i)_{i \in S}$.]

Paper 2, Section II

20H Markov Chains

(i) Let $(X_n)_{n \geqslant 0}$ be a Markov chain on the finite state space S with transition matrix P. Fix a subset $A \subseteq S$, and let

$$H = \inf\{n \geqslant 0 : X_n \in A\}.$$

Fix a function g on S such that $0 < g(i) \leqslant 1$ for all $i \in S$, and let

$$V_i = \mathbb{E}\left[\prod_{n=0}^{H-1} g(X_n) | X_0 = i\right]$$

where $\prod_{n=0}^{-1} a_n = 1$ by convention. Show that

$$V_i = \begin{cases} 1 & \text{if } i \in A \\ g(i) \sum_{j \in S} P_{ij} V_j & \text{otherwise.} \end{cases}$$

(ii) A flea lives on a polyhedron with N vertices, labelled $1, \ldots, N$. It hops from vertex to vertex in the following manner: if one day it is on vertex $i > 1$, the next day it hops to one of the vertices labelled $1, \ldots, i-1$ with equal probability, and it dies upon reaching vertex 1. Let X_n be the position of the flea on day n. What are the transition probabilities for the Markov chain $(X_n)_{n \geqslant 0}$?

(iii) Let H be the number of days the flea is alive, and let

$$V_i = \mathbb{E}(s^H | X_0 = i)$$

where s is a real number such that $0 < s \leqslant 1$. Show that $V_1 = 1$ and

$$\frac{i}{s} V_{i+1} = V_i + \frac{i-1}{s} V_i$$

for $i \geqslant 1$. Conclude that

$$\mathbb{E}(s^H | X_0 = N) = \prod_{i=1}^{N-1} \left(1 + \frac{s-1}{i}\right).$$

[Hint. Use part (i) with $A = \{1\}$ and a well-chosen function g.]

(iv) Show that

$$\mathbb{E}(H | X_0 = N) = \sum_{i=1}^{N-1} \frac{1}{i}.$$

Paper 3, Section I

9E Markov Chains

An intrepid tourist tries to ascend Springfield's famous infinite staircase on an icy day. When he takes a step with his right foot, he reaches the next stair with probability $1/2$, otherwise he falls down and instantly slides back to the bottom with probability $1/2$. Similarly, when he steps with his left foot, he reaches the next stair with probability $1/3$, or slides to the bottom with probability $2/3$. Assume that he always steps first with his right foot when he is at the bottom, and alternates feet as he ascends. Let X_n be his position after his nth step, so that $X_n = i$ when he is on the stair i, $i = 0, 1, 2, \ldots$, where 0 is the bottom stair.

(a) Specify the transition probabilities p_{ij} for the Markov chain $(X_n)_{n \geqslant 0}$ for any $i, j \geqslant 0$.

(b) Find the equilibrium probabilities π_i, for $i \geqslant 0$. [Hint: $\pi_0 = 5/9$.]

(c) Argue that the chain is irreducible and aperiodic and evaluate the limit

$$\lim_{n \to \infty} \mathbb{P}(X_n = i)$$

for each $i \geqslant 0$.

Paper 4, Section I

9E Markov Chains

Consider a Markov chain $(X_n)_{n \geqslant 0}$ with state space $\{a, b, c, d\}$ and transition probabilities given by the following table.

	a	b	c	d
a	1/4	1/4	1/2	0
b	0	1/4	0	3/4
c	1/2	0	1/4	1/4
d	0	1/2	0	1/2

By drawing an appropriate diagram, determine the communicating classes of the chain, and classify them as either open or closed. Compute the following transition and hitting probabilities:

- $\mathbb{P}(X_n = b \mid X_0 = d)$ for a fixed $n \geqslant 0$,
- $\mathbb{P}(X_n = c \text{ for some } n \geqslant 1 \mid X_0 = a)$.

Paper 1, Section II

20E Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a Markov chain.

(a) What does it mean to say that a state i is positive recurrent? How is this property related to the equilibrium probability π_i? You do not need to give a full proof, but you should carefully state any theorems you use.

(b) What is a communicating class? Prove that if states i and j are in the same communicating class and i is positive recurrent then j is positive recurrent also.

A frog is in a pond with an infinite number of lily pads, numbered $1, 2, 3, \ldots$. She hops from pad to pad in the following manner: if she happens to be on pad i at a given time, she hops to one of pads $(1, 2, \ldots, i, i+1)$ with equal probability.

(c) Find the equilibrium distribution of the corresponding Markov chain.

(d) Now suppose the frog starts on pad k and stops when she returns to it. Show that the expected number of times the frog hops is $e(k-1)!$ where $e = 2.718\ldots$. What is the expected number of times she will visit the lily pad $k+1$?

Paper 2, Section II

20E Markov Chains

Let $(X_n)_{n\geqslant 0}$ be a simple, symmetric random walk on the integers $\{\ldots, -1, 0, 1, \ldots\}$, with $X_0 = 0$ and $\mathbb{P}(X_{n+1} = i \pm 1 \,|\, X_n = i) = 1/2$. For each integer $a \geqslant 1$, let $T_a = \inf\{n \geqslant 0 : X_n = a\}$. Show that T_a is a stopping time.

Define a random variable Y_n by the rule

$$Y_n = \begin{cases} X_n & \text{if } n < T_a\,, \\ 2a - X_n & \text{if } n \geqslant T_a\,. \end{cases}$$

Show that $(Y_n)_{n\geqslant 0}$ is also a simple, symmetric random walk.

Let $M_n = \max_{0\leqslant i\leqslant n} X_n$. Explain why $\{M_n \geqslant a\} = \{T_a \leqslant n\}$ for $a \geqslant 0$. By using the process $(Y_n)_{n\geqslant 0}$ constructed above, show that, for $a \geqslant 0$,

$$\mathbb{P}(M_n \geqslant a,\, X_n \leqslant a-1) = \mathbb{P}(X_n \geqslant a+1),$$

and thus

$$\mathbb{P}(M_n \geqslant a) = \mathbb{P}(X_n \geqslant a) + \mathbb{P}(X_n \geqslant a+1).$$

Hence compute

$$\mathbb{P}(M_n = a)$$

when a and n are positive integers with $n \geqslant a$. [Hint: if n is even, then X_n must be even, and if n is odd, then X_n must be odd.]

Paper 3, Section I

9H Markov Chains

Let $(X_n)_{n \geqslant 0}$ be a simple random walk on the integers: the random variables $\xi_n \equiv X_n - X_{n-1}$ are independent, with distribution

$$P(\xi = 1) = p, \quad P(\xi = -1) = q,$$

where $0 < p < 1$, and $q = 1-p$. Consider the hitting time $\tau = \inf\{n : X_n = 0 \text{ or } X_n = N\}$, where $N > 1$ is a given integer. For fixed $s \in (0, 1)$ define $\xi_k = E[s^\tau : X_\tau = 0 | X_0 = k]$ for $k = 0, \ldots, N$. Show that the ξ_k satisfy a second-order difference equation, and hence find them.

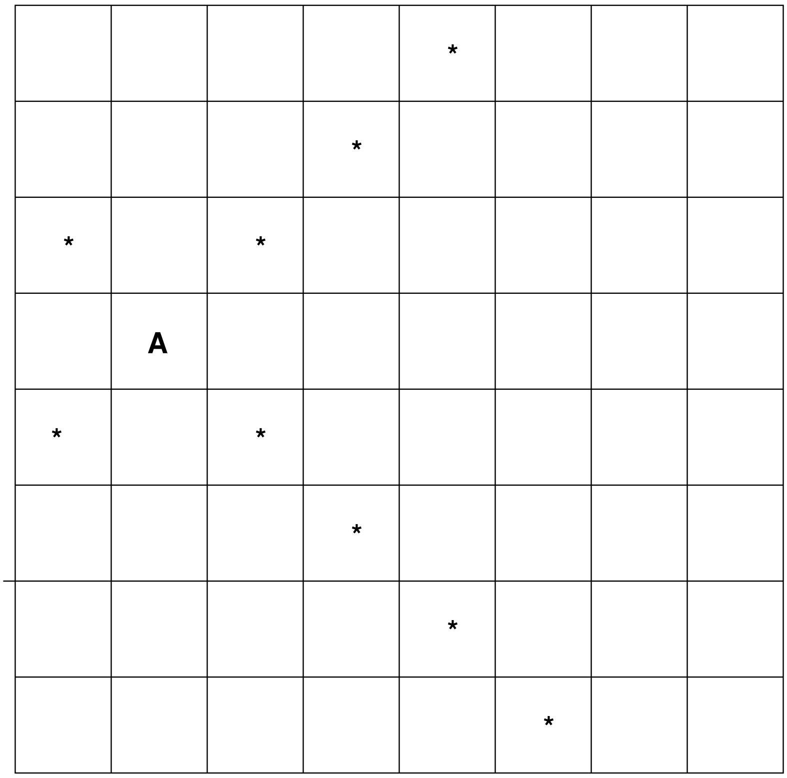
Paper 4, Section I

9H Markov Chains

In chess, a bishop is allowed to move only in straight diagonal lines. Thus if the bishop stands on the square marked A in the diagram, it is able in one move to reach any of the squares marked with an asterisk. Suppose that the bishop moves at random around the chess board, choosing at each move with equal probability from the squares it can reach, the square chosen being independent of all previous choices. The bishop starts at the bottom left-hand corner of the board.

If X_n is the position of the bishop at time n, show that $(X_n)_{n\geqslant 0}$ is a reversible Markov chain, whose statespace you should specify. Find the invariant distribution of this Markov chain.

What is the expected number of moves the bishop will make before first returning to its starting square?

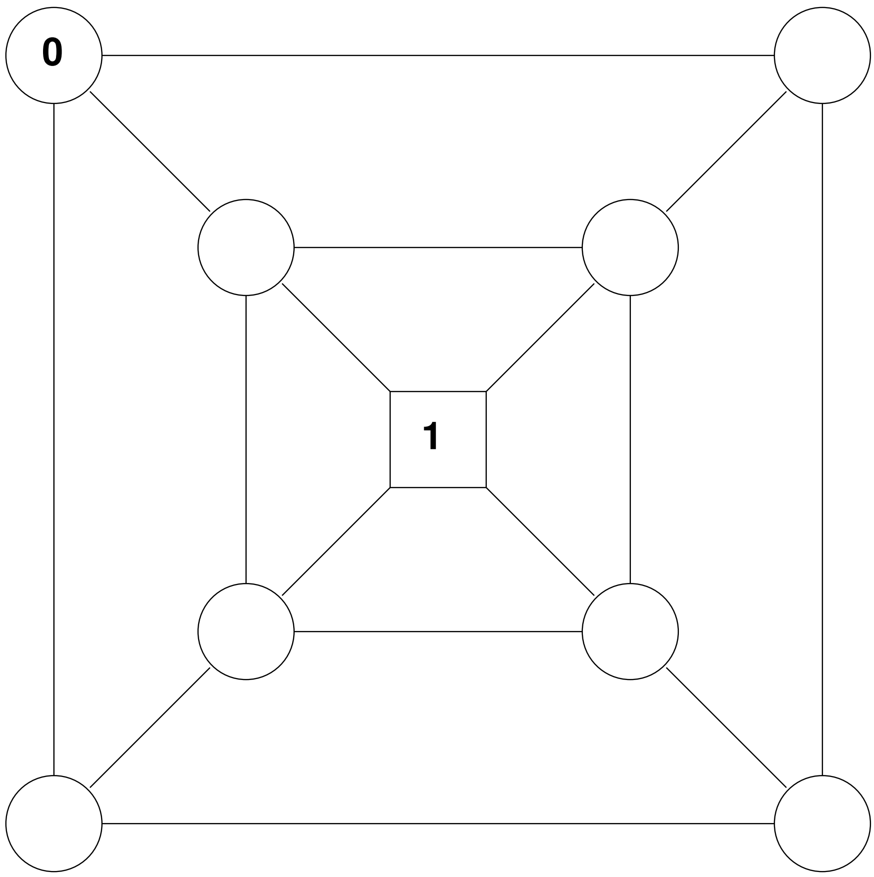


Paper 1, Section II

19H Markov Chains

A gerbil is introduced into a maze at the node labelled 0 in the diagram. It roams at random through the maze until it reaches the node labelled 1. At each vertex, it chooses to move to one of the neighbouring nodes with equal probability, independently of all other choices. Find the mean number of moves required for the gerbil to reach node 1.

Suppose now that the gerbil is intelligent, in that when it reaches a node it will not immediately return to the node from which it has just come, choosing with equal probability from all other neighbouring nodes. Express the movement of the gerbil in terms of a Markov chain whose states and transition probabilities you should specify. Find the mean number of moves until the intelligent gerbil reaches node 1. Compare with your answer to the first part, and comment briefly.



Paper 2, Section II

20H Markov Chains

Suppose that B is a non-empty subset of the statespace I of a Markov chain X with transition matrix P, and let $\tau \equiv \inf\{n \geqslant 0 : X_n \in B\}$, with the convention that $\inf \emptyset = \infty$. If $h_i = P(\tau < \infty | X_0 = i)$, show that the equations

$$(a) \qquad g_i \geqslant (Pg)_i \equiv \sum_{j \in I} p_{ij} g_j \geqslant 0 \quad \forall i,$$

$$(b) \qquad g_i = 1 \quad \forall i \in B$$

are satisfied by $g = h$.

If g satisfies (a), prove that g also satisfies

$$(c) \qquad g_i \geqslant (\tilde{P} g)_i \quad \forall i,$$

where

$$\tilde{p}_{ij} = \begin{cases} p_{ij} & (i \notin B), \\ \delta_{ij} & (i \in B). \end{cases}$$

By interpreting the transition matrix $\tilde{P}$, prove that h is the minimal solution to the equations (a), (b).

Now suppose that P is irreducible. Prove that P is recurrent if and only if the only solutions to (a) are constant functions.

1/II/19H Markov Chains

The village green is ringed by a fence with N fenceposts, labelled $0, 1, \ldots, N-1$. The village idiot is given a pot of paint and a brush, and started at post 0 with instructions to paint all the posts. He paints post 0, and then chooses one of the two nearest neighbours, 1 or $N-1$, with equal probability, moving to the chosen post and painting it. After painting a post, he chooses with equal probability one of the two nearest neighbours, moves there and paints it (regardless of whether it is already painted). Find the distribution of the last post unpainted.

2/II/20H Markov Chains

A Markov chain with state–space $I = \mathbb{Z}^+$ has non-zero transition probabilities $p_{00} = q_0$ and

$$p_{i,i+1} = p_i\,, \quad p_{i+1,i} = q_{i+1} \qquad (i \in I)\,.$$

Prove that this chain is recurrent if and only if

$$\sum_{n \geqslant 1} \prod_{r=1}^{n} \frac{q_r}{p_r} = \infty\,.$$

Prove that this chain is positive-recurrent if and only if

$$\sum_{n \geqslant 1} \prod_{r=1}^{n} \frac{p_{r-1}}{q_r} < \infty\,.$$

3/I/9H Markov Chains

What does it mean to say that a Markov chain is recurrent?

Stating clearly any general results to which you appeal, prove that the symmetric simple random walk on $\mathbb{Z}$ is recurrent.

4/I/9H **Markov Chains**

A Markov chain on the state–space $I = \{1, 2, 3, 4, 5, 6, 7\}$ has transition matrix

$$P = \begin{pmatrix} 0 & 1/2 & 1/4 & 0 & 1/4 & 0 & 0 \\ 1/3 & 0 & 1/2 & 0 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 1/2 \end{pmatrix}.$$

Classify the chain into its communicating classes, deciding for each what the period is, and whether the class is recurrent.

For each $i, j \in I$ say whether the limit $\lim_{n\to\infty} p_{ij}^{(n)}$ exists, and evaluate the limit when it does exist.

1/II/19C Markov Chains

Consider a Markov chain $(X_n)_{n \geqslant 0}$ on states $\{0, 1, \ldots, r\}$ with transition matrix (P_{ij}), where $P_{0,0} = 1 = P_{r,r}$, so that 0 and r are absorbing states. Let

$$A = (X_n = 0, \text{ for some } n \geqslant 0) ,$$

be the event that the chain is absorbed in 0. Assume that $h_i = \mathbb{P}(A \mid X_0 = i) > 0$ for $1 \leqslant i < r$.

Show carefully that, conditional on the event A, $(X_n)_{n \geqslant 0}$ is a Markov chain and determine its transition matrix.

Now consider the case where $P_{i,i+1} = \frac{1}{2} = P_{i,i-1}$, for $1 \leqslant i < r$. Suppose that $X_0 = i$, $1 \leqslant i < r$, and that the event A occurs; calculate the expected number of transitions until the chain is first in the state 0.

2/II/20C Markov Chains

Consider a Markov chain with state space $S = \{0, 1, 2, \ldots\}$ and transition matrix given by

$$P_{i,j} = \begin{cases} qp^{j-i+1} & \text{for } i \geqslant 1 \text{ and } j \geqslant i-1, \\ qp^j & \text{for } i = 0 \text{ and } j \geqslant 0, \end{cases}$$

and $P_{i,j} = 0$ otherwise, where $0 < p = 1 - q < 1$.

For each value of p, $0 < p < 1$, determine whether the chain is transient, null recurrent or positive recurrent, and in the last case find the invariant distribution.

3/I/9C Markov Chains

Consider a Markov chain $(X_n)_{n \geqslant 0}$ with state space $S = \{0, 1\}$ and transition matrix

$$P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix} ,$$

where $0 < \alpha < 1$ and $0 < \beta < 1$.

Calculate $\mathbb{P}(X_n = 0 \mid X_0 = 0)$ for each $n \geqslant 0$.

4/I/9C Markov Chains

For a Markov chain with state space S, define what is meant by the following:

(i) states $i, j \in S$ *communicate*;

(ii) state $i \in S$ is *recurrent*.

Prove that communication is an equivalence relation on S and that if two states i, j communicate and i is recurrent then j is recurrent.

1/II/19C **Markov Chains**

Explain what is meant by a stopping time of a Markov chain $(X_n)_{n\geq 0}$. State the strong Markov property.

Show that, for any state i, the probability, starting from i, that $(X_n)_{n\geq 0}$ makes infinitely many visits to i can take only the values 0 or 1.

Show moreover that, if

$$\sum_{n=0}^{\infty} \mathbb{P}_i(X_n = i) = \infty,$$

then $(X_n)_{n\geq 0}$ makes infinitely many visits to i with probability 1.

2/II/20C **Markov Chains**

Consider the Markov chain $(X_n)_{n\geq 0}$ on the integers $\mathbb{Z}$ whose non-zero transition probabilities are given by $p_{0,1} = p_{0,-1} = 1/2$ and

$$p_{n,n-1} = 1/3, \quad p_{n,n+1} = 2/3, \qquad \text{for } n \geq 1,$$

$$p_{n,n-1} = 3/4, \quad p_{n,n+1} = 1/4, \qquad \text{for } n \leqslant -1.$$

(a) Show that, if $X_0 = 1$, then $(X_n)_{n\geq 0}$ hits 0 with probability 1/2.

(b) Suppose now that $X_0 = 0$. Show that, with probability 1, as $n \to \infty$ either $X_n \to \infty$ or $X_n \to -\infty$.

(c) In the case $X_0 = 0$ compute $\mathbb{P}(X_n \to \infty \text{ as } n \to \infty)$.

3/I/9C **Markov Chains**

A hungry student always chooses one of three places to get his lunch, basing his choice for one day on his gastronomic experience the day before. He sometimes tries a sandwich from Natasha's Patisserie: with probability 1/2 this is delicious so he returns the next day; if the sandwich is less than delicious, he chooses with equal probability 1/4 either to eat in Hall or to cook for himself. Food in Hall leaves no strong impression, so he chooses the next day each of the options with equal probability 1/3. However, since he is a hopeless cook, he never tries his own cooking two days running, always preferring to buy a sandwich the next day. On the first day of term the student has lunch in Hall. What is the probability that 60 days later he is again having lunch in Hall?

[*Note* $0^0 = 1$.]

4/I/9C **Markov Chains**

A game of chance is played as follows. At each turn the player tosses a coin, which lands heads or tails with equal probability 1/2. The outcome determines a score for that turn, which depends also on the cumulative score so far. Write S_n for the cumulative score after n turns. In particular $S_0 = 0$. When S_n is odd, a head scores 1 but a tail scores 0. When S_n is a multiple of 4, a head scores 4 and a tail scores 1. When S_n is even but is not a multiple of 4, a head scores 2 and a tail scores 1. By considering a suitable four-state Markov chain, determine the long run proportion of turns for which S_n is a multiple of 4. State clearly any general theorems to which you appeal.

1/II/19D **Markov Chains**

Every night Lancelot and Guinevere sit down with four guests for a meal at a circular dining table. The six diners are equally spaced around the table and just before each meal two individuals are chosen at random and they exchange places from the previous night while the other four diners stay in the same places they occupied at the last meal; the choices on successive nights are made independently. On the first night Lancelot and Guinevere are seated next to each other.

Find the probability that they are seated diametrically opposite each other on the $(n+1)$th night at the round table, $n \geqslant 1$.

2/II/20D **Markov Chains**

Consider a Markov chain $(X_n)_{n\geqslant 0}$ with state space $\{0,1,2,\ldots\}$ and transition probabilities given by

$$P_{i,j} = pq^{i-j+1}, \quad 0 < j \leqslant i+1, \quad \text{and} \quad P_{i,0} = q^{i+1} \quad \text{for} \quad i \geqslant 0,$$

with $P_{i,j} = 0$, otherwise, where $0 < p < 1$ and $q = 1 - p$.

For each $i \geqslant 1$, let

$$h_i = \mathbb{P}\left(X_n = 0, \text{for some } n \geqslant 0 \mid X_0 = i\right),$$

that is, the probability that the chain ever hits the state 0 given that it starts in state i. Write down the equations satisfied by the probabilities $\{h_i,\, i \geqslant 1\}$ and hence, or otherwise, show that they satisfy a second-order recurrence relation with constant coefficients. Calculate h_i for each $i \geqslant 1$.

Determine for each value of p, $0 < p < 1$, whether the chain is transient, null recurrent or positive recurrent and in the last case calculate the stationary distribution.

[*Hint: When the chain is positive recurrent, the stationary distribution is geometric.*]

3/I/9D **Markov Chains**

Prove that if two states of a Markov chain communicate then they have the same period.

Consider a Markov chain with state space $\{1, 2, \ldots, 7\}$ and transition probabilities determined by the matrix

$$\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Identify the communicating classes of the chain and for each class state whether it is open or closed and determine its period.

4/I/9D **Markov Chains**

Prove that the simple symmetric random walk in three dimensions is transient.

[*You may wish to recall Stirling's formula:* $n! \sim (2\pi)^{\frac{1}{2}}\, n^{n+\frac{1}{2}} e^{-n}$.]

1/I/11H Markov Chains

Let $P = (P_{ij})$ be a transition matrix. What does it mean to say that P is (a) irreducible, (b) recurrent?

Suppose that P is irreducible and recurrent and that the state space contains at least two states. Define a new transition matrix $\tilde{P}$ by

$$\tilde{P}_{ij} = \begin{cases} 0 & \text{if} \quad i = j, \\ (1 - P_{ii})^{-1} P_{ij} & \text{if} \quad i \neq j. \end{cases}$$

Prove that $\tilde{P}$ is also irreducible and recurrent.

1/II/22H Markov Chains

Consider the Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$ and transition matrix

$$\begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Determine the communicating classes of the chain, and for each class indicate whether it is open or closed.

Suppose that the chain starts in state 2; determine the probability that it ever reaches state 6.

Suppose that the chain starts in state 3; determine the probability that it is in state 6 after exactly n transitions, $n \geqslant 1$.

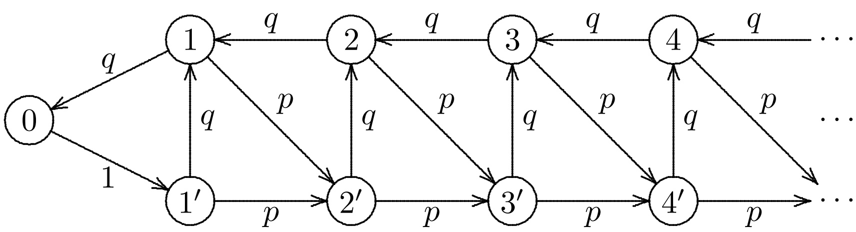
2/I/11H Markov Chains

Let $(X_r)_{r \geqslant 0}$ be an irreducible, positive-recurrent Markov chain on the state space S with transition matrix (P_{ij}) and initial distribution $P(X_0 = i) = \pi_i$, $i \in S$, where (π_i) is the unique invariant distribution. What does it mean to say that the Markov chain is reversible?

Prove that the Markov chain is reversible if and only if $\pi_i P_{ij} = \pi_j P_{ji}$ for all $i, j \in S$.

2/II/22H **Markov Chains**

Consider a Markov chain on the state space $S = \{0, 1, 2, \ldots\} \cup \{1', 2', 3', \ldots\}$ with transition probabilities as illustrated in the diagram below, where $0 < q < 1$ and $p = 1 - q$.



For each value of q, $0 < q < 1$, determine whether the chain is transient, null recurrent or positive recurrent.

When the chain is positive recurrent, calculate the invariant distribution.

A1/1 B1/1 **Markov Chains**

(i) Let $(X_n, Y_n)_{n\geqslant 0}$ be a simple symmetric random walk in $\mathbb{Z}^2$, starting from $(0,0)$, and set $T = \inf\{n \geqslant 0 : \max\{|X_n|, |Y_n|\} = 2\}$. Determine the quantities $\mathbb{E}(T)$ and $\mathbb{P}(X_T = 2 \text{ and } Y_T = 0)$.

(ii) Let $(X_n)_{n\geqslant 0}$ be a discrete-time Markov chain with state-space I and transition matrix P. What does it mean to say that a state $i \in I$ is recurrent? Prove that i is recurrent if and only if $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$, where $p_{ii}^{(n)}$ denotes the (i,i) entry in P^n.

Show that the simple symmetric random walk in $\mathbb{Z}^2$ is recurrent.

A2/1 **Markov Chains**

~~**(i)** What is meant by a Poisson process of rate λ? Show that if $(X_t)_{t\geqslant 0}$ and $(Y_t)_{t\geqslant 0}$ are independent Poisson processes of rates λ and μ respectively, then $(X_t + Y_t)_{t\geqslant 0}$ is also a Poisson process, and determine its rate.~~

~~**(ii)** A Poisson process of rate λ is observed by someone who believes that the first holding time is longer than all subsequent holding times. How long on average will it take before the observer is proved wrong?~~

A3/1 B3/1 **Markov Chains**

~~**(i)** Consider the continuous-time Markov chain $(X_t)_{t\geqslant 0}$ with state-space $\{1,2,3,4\}$ and Q-matrix~~

$$Q = \begin{pmatrix} -2 & 0 & 0 & 2 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 1 & 5 & 2 & -8 \end{pmatrix}.$$

~~Set~~

$$Y_t = \begin{cases} X_t & \text{if } X_t \in \{1,2,3\} \\ 2 & \text{if } X_t = 4 \end{cases}$$

~~and~~

$$Z_t = \begin{cases} X_t & \text{if } X_t \in \{1,2,3\} \\ 1 & \text{if } X_t = 4. \end{cases}$$

~~Determine which, if any, of the processes $(Y_t)_{t\geqslant 0}$ and $(Z_t)_{t\geqslant 0}$ are Markov chains.~~

~~**(ii)** Find an invariant distribution for the chain $(X_t)_{t\geqslant 0}$ given in Part (i). Suppose $X_0 = 1$. Find, for all $t \geqslant 0$, the probability that $X_t = 1$.~~

A4/1 **Markov Chains**

Consider a pack of cards labelled $1, \ldots, 52$. We repeatedly take the top card and insert it uniformly at random in one of the 52 possible places, that is, either on the top or on the bottom or in one of the 50 places inside the pack. How long on average will it take for the bottom card to reach the top?

Let p_n denote the probability that after n iterations the cards are found in increasing order. Show that, irrespective of the initial ordering, p_n converges as $n \to \infty$, and determine the limit p. You should give precise statements of any general results to which you appeal.

Show that, at least until the bottom card reaches the top, the ordering of cards inserted beneath it is uniformly random. Hence or otherwise show that, for all n,

$$|p_n - p| \leqslant 52(1 + \log 52)/n \ .$$

A1/1 B1/1 **Markov Chains**

(i) We are given a finite set of airports. Assume that between any two airports, i and j, there are $a_{ij} = a_{ji}$ flights in each direction on every day. A confused traveller takes one flight per day, choosing at random from all available flights. Starting from i, how many days on average will pass until the traveller returns again to i? Be careful to allow for the case where there may be no flights at all between two given airports.

(ii) Consider the infinite tree T with root R, where, for all $m \geqslant 0$, all vertices at distance 2^m from R have degree 3, and where all other vertices (except R) have degree 2. Show that the random walk on T is recurrent.

A2/1 **Markov Chains**

~~**(i)** In each of the following cases, the state-space I and non-zero transition rates q_{ij} ($i \neq j$) of a continuous-time Markov chain are given. Determine in which cases the chain is explosive.~~

~~(a)~~ $\cancel{I = \{1, 2, 3, \ldots\}}, \quad \cancel{q_{i,i+1} = i^2}, \quad \cancel{i \in I},$

~~(b)~~ $\cancel{I = \mathbb{Z}}, \quad \cancel{q_{i,i-1} = q_{i,i+1} = 2^i}, \quad \cancel{i \in I}.$

~~**(ii)** Children arrive at a see-saw according to a Poisson process of rate 1. Initially there are no children. The first child to arrive waits at the see-saw. When the second child arrives, they play on the see-saw. When the third child arrives, they all decide to go and play on the merry-go-round. The cycle then repeats. Show that the number of children at the see-saw evolves as a Markov Chain and determine its generator matrix. Find the probability that there are no children at the see-saw at time t.~~

~~Hence obtain the identity~~

$$\cancel{\sum_{n=0}^{\infty} e^{-t} \frac{t^{3n}}{(3n)!} = \frac{1}{3} + \frac{2}{3}\, e^{-\frac{3}{2}t} \cos \frac{\sqrt{3}}{2} t \,.}$$

A3/1 B3/1 **Markov Chains**

~~**(i)** Consider the continuous-time Markov chain~~ $(X_t)_{t\geqslant 0}$ ~~on~~ $\{1,2,3,4,5,6,7\}$ ~~with generator matrix~~

$$Q=\begin{pmatrix} -6 & 2 & 0 & 0 & 0 & 4 & 0 \\ 2 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -5 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & -6 & 0 & 2 \\ 1 & 2 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

~~Compute the probability, starting from state 3, that~~ X_t ~~hits state 2 eventually.~~

~~Deduce that~~

$$\lim_{t\to\infty}\mathbb{P}(X_t=2|X_0=3)=\frac{4}{15}.$$

~~[*Justification of standard arguments is not expected.*]~~

~~**(ii)** A colony of cells contains immature and mature cells. Each immature cell, after an exponential time of parameter 2, becomes a mature cell. Each mature cell, after an exponential time of parameter 3, divides into two immature cells. Suppose we begin with one immature cell and let~~ $n(t)$ ~~denote the expected number of immature cells at time~~ t. ~~Show that~~

$$n(t)=(4e^t+3e^{-6t})/7.$$

A4/1 **Markov Chains**

Write an essay on the long-time behaviour of discrete-time Markov chains on a finite state space. Your essay should include discussion of the convergence of probabilities as well as almost-sure behaviour. You should also explain what happens when the chain is not irreducible.

A1/1 B1/1 **Markov Chains**

(i) Let $X = (X_n : 0 \leqslant n \leqslant N)$ be an irreducible Markov chain on the finite state space S with transition matrix $P = (p_{ij})$ and invariant distribution π. What does it mean to say that X is reversible in equilibrium?

Show that X is reversible in equilibrium if and only if $\pi_i p_{ij} = \pi_j p_{ji}$ for all $i, j \in S$.

(ii) A finite connected graph G has vertex set V and edge set E, and has neither loops nor multiple edges. A particle performs a random walk on V, moving at each step to a randomly chosen neighbour of the current position, each such neighbour being picked with equal probability, independently of all previous moves. Show that the unique invariant distribution is given by $\pi_v = d_v/(2|E|)$ where d_v is the degree of vertex v.

A rook performs a random walk on a chessboard; at each step, it is equally likely to make any of the moves which are legal for a rook. What is the mean recurrence time of a corner square. (You should give a clear statement of any general theorem used.)

[*A chessboard is an* 8×8 *square grid. A legal move is one of any length parallel to the axes.*]

A2/1 **Markov Chains**

~~**(i)** The fire alarm in Mill Lane is set off at random times. The probability of an alarm during the time-interval $(u, u+h)$ is $\lambda(u)h + o(h)$ where the 'intensity function' $\lambda(u)$ may vary with the time u. Let $N(t)$ be the number of alarms by time t, and set $N(0) = 0$. Show, subject to reasonable extra assumptions to be stated clearly, that $p_i(t) = P(N(t) = i)$ satisfies~~

$$\cancel{p_0'(t) = -\lambda(t)p_0(t), \quad p_i'(t) = \lambda(t)\{p_{i-1}(t) - p_i(t)\}, \quad i \geqslant 1.}$$

~~Deduce that $N(t)$ has the Poisson distribution with parameter $\Lambda(t) = \int_0^t \lambda(u)du$.~~

~~**(ii)** The fire alarm in Clarkson Road is different. The number $M(t)$ of alarms by time t is such that~~

$$\cancel{P(M(t+h) = m+1 \mid M(t) = m) = \lambda_m h + o(h)},$$

~~where $\lambda_m = \alpha m + \beta$, $m \geqslant 0$, and $\alpha, \beta > 0$. Show, subject to suitable extra conditions, that $p_m(t) = P(M(t) = m)$ satisfies a set of differential-difference equations to be specified. Deduce without solving these equations in their entirety that $M(t)$ has mean $\beta(e^{\alpha t} - 1)/\alpha$, and find the variance of $M(t)$.~~

~~A3/1 B3/1 **Markov Chains**~~

~~**(i)** Explain what is meant by the *transition semigroup* $\{P_t\}$ of a Markov chain X in continuous time. If the state space is finite, show under assumptions to be stated clearly, that $P_t' = GP_t$ for some matrix G. Show that a distribution π satisfies $\pi G = 0$ if and only if $\pi P_t = \pi$ for all $t \geqslant 0$, and explain the importance of such π.~~

~~**(ii)** Let X be a continuous-time Markov chain on the state space $S = \{1, 2\}$ with generator~~

$$G = \begin{pmatrix} -\beta & \beta \\ \gamma & -\gamma \end{pmatrix}, \qquad \text{where} \quad \beta, \gamma > 0.$$

~~Show that the transition semigroup $P_t = \exp(tG)$ is given by~~

$$(\beta + \gamma)P_t = \begin{pmatrix} \gamma + \beta h(t) & \beta(1 - h(t)) \\ \gamma(1 - h(t)) & \beta + \gamma h(t) \end{pmatrix},$$

~~where $h(t) = e^{-t(\beta+\gamma)}$.~~

~~For $0 < \alpha < 1$, let~~

$$H(\alpha) = \begin{pmatrix} \alpha & 1 - \alpha \\ 1 - \alpha & \alpha \end{pmatrix}.$$

~~For a continuous-time chain X, let M be a matrix with (i, j) entry $P(X(1) = j \mid X(0) = i)$, for $i, j \in S$. Show that there is a chain X with $M = H(\alpha)$ if and only if $\alpha > \frac{1}{2}$.~~

A4/1 **Markov Chains**

Write an essay on the convergence to equilibrium of a discrete-time Markov chain on a countable state-space. You should include a discussion of the existence of invariant distributions, and of the limit theorem in the non-null recurrent case.