Markov Methods Review

$i\ j$ are in the same class if $p_{ij}^M p_{ji}^N > 0$ some $M\ N$.
i is persistant if $pr_i(\ i$ hit some $t > 0\ ) = 1$
$f_{ij}^n = pr_i(\ $no j hit $0<t<n\ ,\ X(n) = j)$ $\qquad F_{ii} = \sum_1^\infty f_{ii}^n = 1$ iff i persistant.
using $Y(n) = I[X(t) = n]$ we get $E_i(\#\text{ hits on } i) = \sum_1^\infty p_{ii}^n$.

Persistance is a class property : $\quad 1 = pr_i(\infty \text{ hits on } i) = \sum_h p_{ih}^M p_h(\infty \text{ hits on } i)$
a persistant class is closed : $\quad \sum p_{ih}^M = 1$ so $p_{ih}^M > 0 \Rightarrow p_h(\infty \text{ hits on } i) = 1$
a finite closed class is persistant : $\quad$ let $A_{ij} = [$starts at i hits j ∞ often$]$
$pr_i(A_{ij}) > 0$ some $j \Rightarrow pr_j(j \text{ hit i.o.}) = 1$.

C a closed class, $\alpha_{iC} = pr_i(\text{ultimately trapped in } C)$ then
$P\alpha_C = \alpha_C \qquad \alpha_{iC} = 1$ if $i \in C$ and $a_{iC} \le 1$
- $Py = y \qquad y_i = 1$ on $C \quad 0 \le y_i \le 1$ are the RHE and α_C is their min. soln.
  $(\ y_i \ge \sum_C p_{ih}^n = pr_i(\text{absorbed in } C \text{ at } n\text{th step})$ let $n \to \infty\ )$.

Single class statespace

- $x \ge 0 \quad (xP)_I \le x_I = 1 \quad (xP)_j = x_j \quad j \ne I$ have a soln with

$(xP)_I < x_I$ iff class is transient. These eqn^s always have a minimal sol^n z s.t. $\forall\ x \geq 0$ s.t. $xP \leq x$

$x \geq x_I z$ transient $= x_I z$ persistant

pf:

1. Consider any $x \geq 0$ $xP \leq x$ $x_i p_{ij} \leq x_j$, substitute into self to get
$x_i p_{ij} + \sum_{h \neq i} x_i p_{ih} p_{hj} + \sum_{h \neq i} \sum_{k \neq i} x_i p_{ih} p_{hk} p_{kj} \leq x_j$ etc.
$x_i E_i(\#\, j$ hits not preceded by an i hit$) \leq x_j$..
2. Let $N_{ij} =$ ↑. $N_{ii} = F_{ii} \leq 1$ take $i \neq j$
$f_{ii}^{M+n} \geq$ $pr_i($no i hit save $X(M+n) = i$ and $X(n) = j)$
$=$ $pr_i(\, i$ avoided $0 < t < n$ $X(n) = j.)\, f_{ji}^{M}$
Sum to get $1 \geq F_{ii} \geq \sum_{r \geq M} f_{ii}^{r} \geq N_{ij} f_{ji}^{M}$. giving N_{ij} finite.
3. Now put $z_{ii} = 1$ $z_{ij} = N_{ij}$ let $z_j = z_{Ij}$
$(zP)_j = p_{Ij} + \sum_{n > 1} pr_I($no I hit $0 < t < n$ $X(n) = j)$
$= z_j \quad j \neq I$
$= F_{II} \quad j = I \quad \substack{< \\ =}\ 1$ as $\substack{\text{trans.} \\ \text{pers.}}$
and $x_j \geq x_I z_{Ij} = z_j$ all x with $xP \leq x$ $x_I = 1$. so minimal.
4. Show that in the persistant case $z_{ij} z_{ji} = 1$. <u>So all $z_{ij} > 0$</u>
$\therefore$ $x_j \geq x_i z_{ij} \geq x_j z_{ji} z_{ij} = x_j \Rightarrow x_j = x_i z_{ij}$ $z_{ij} = x_j / x_i$

Ergodic Th^m

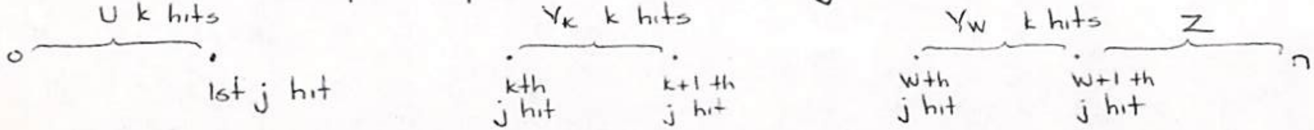
- all states a persistant class. $H_n(k|i)/H_n(j|i) \to \mu_k/\mu_j$

where μ is the sol^n to $\mu \geq 0$ $\mu P = \mu$.

pf:

$H_n(j|i)$ = # j hits in $0 < t \leq n$ for a path starting in i.

take a sample path $W = H_n(j|i) - 1 \geq 1$



$$\frac{H_n(k|i)}{H_n(j|i)} = \underbrace{\frac{U}{W+1}}_{\to 0} + \underbrace{\frac{Y_1 + \cdots + Y_W}{W+1}}_{\to E(Y) \text{ by SLLN } = N_{jk} = z_{jk} = \mu_k/\mu_j} + \frac{Z}{W+1} \to 0.$$

Behavior of P_{ij}^n.

- $p_{ij}^{n+1} = f_{ij}^{n+1} + (f_{ij}^n p_{jj} + \cdots + f_{ij} p_{jj}^n)$. $\quad \sum p_{ij}^n s^n = P_{ij}(s)$ $\quad \sum f_{ij}^n s^n = F_{ij}(s)$

$P_{ij}(s) = F_{ij}(s) + F_{ij}(s) P_{jj}(s) = F_{ij}(s)/(1 - F_{jj}(s))$

$A \lim p_{ij}^n = F_{ij} \lim_{s \to 1} (1-s)/(1 - F_{jj}(s)) \to 0$ unless $F_{jj} = 1$

then $\dfrac{1 - F_{jj}(s)}{1-s} = \sum_{n=1} f_{jj}^n (1 + s + \cdots + s^{n-1}) = \sum_1^\infty n f_{jj}^n = m_j$

$\therefore A \lim p_{ij}^n = F_{ij}/m_j$

$m_j = E_j$(time to first hit on j or ∞ if no hit) = mean recurrance time

$\therefore$ $A\text{-}\lim p_{ij}^n = \alpha_{i,C(j)}/m_j = \pi_{ij}$

<u>one persistant class:</u>

$Q(s) = \sum P^n s^n \qquad R(s) = (1-s)Q(s)$

$sPR(s) = sR(s)P = R(s) - s(1-s)P \qquad R(s) \to \Pi$ as $s \to 1$

$\pi_{ij} = 1/m_j \qquad \Pi = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}(1/m_1, \dots) = 1.V$

$\Pi P \leq \Pi \qquad P\Pi = \Pi$

so $1.VP \leq 1.V \qquad VP \leq V \quad \therefore V = (1/m_1, \dots)$ is controlled by our invariant measure $xP = x$

$\therefore$ $\Pi = 0$ or. identical strictly +tive rows

NULL POSITIVE.

positive case, $\Pi R(s) = R(s)\Pi = \Pi$ since $\Pi P = P\Pi = \Pi$

so $\Pi^2 = \Pi \qquad 1.V1.V = 1.V \Rightarrow V1 = 1 \qquad \sum 1/m_j = 1$

$\therefore$ In the case of a single +tive persistant class,

$\exists\ \mu > 0$ s.t $\ni \mu P = P\mu = \mu$ and $\sum \mu_j < \infty \qquad \pi_{ij} = \mu_j / \sum \mu_k$

so $\sum_j \pi_{ij} = 1$

and $\sum \mu_j = \infty$ in the null case and $\pi_{ij} = 0$.

Continuous Time

$q_{ij} = p_{ij}'(0)$ always exist $\quad p_{ij}(t) = 1 - q_i t + o(t) \quad j = i$

$\qquad\qquad\qquad\qquad\qquad\qquad = q_{ij} t + o(t) \quad i \neq j$

if $0 \leq q_i \leq M < \infty$ all i $\quad P_t = \exp Qt \quad$ include finite state case.

<u>ABC conditions</u>:

(A) each q_i is finite $\quad$ (B) $\sum_{j \neq i} q_{ij} = q_i \quad \Rightarrow \exists$ MP with this Q

(C) $Qy = y \quad y_i < M < \infty$ all $i \quad \Rightarrow$ this MP is unique

(includes finite state case and $0 \leq q_i \leq M$ case).

<u>$p_{ii}(t) > 0$ all t</u> : consider $p_{ii}(t) \geq (p_{ii}(t/n))^n \quad p_{ii}(t/n) \to 1$ as $n \to \infty$.

$$\begin{aligned} p_{ij}(t+\delta) - p_{ij}(t) &= \sum_{\alpha \neq i} p_{i\alpha}(\delta) p_{\alpha j}(t) + p_{ii}(\delta) p_{ij}(t) - p_{ij}(t) \\ &\leq \sum_{\alpha \neq i} p_{i\alpha}(\delta) + \text{—} = 1 - p_{ii}(\delta) + -p_{ij}(t)(1 - p_{ii}(\delta)) \\ &= (1 - p_{ij}(t))(1 - p_{ii}(\delta)) \end{aligned}$$

also $\geq -p_{ij}(t)(1 - p_{ii}(\delta)) \quad \therefore \quad |p_{ij}(t+\delta) - p_{ij}(t)| \leq 1 - p_{ii}(\delta)$

$\Rightarrow$ <u>$p_{ij}(t)$ is uniformly cts.</u>

residence times:

pr_i(stays in i up to time t) $= \lim_{n \to \infty} \mathrm{pr}_i$(in i at all $mt/2^n$)

$$= \lim_{n \to \infty} \{ p_{ii}(t/2^n) \}^{2^n + 1} = \lim_{n \to \infty} \left\{ 1 - \frac{t}{2^n} \frac{(1 - p_{ii}(t/2^n))}{t/2^n} \right\}^{2^n + 1} = e^{-q_i t}$$

$\therefore$ p_i(jump from i before time t) = $1 - e^{-q_i t}$

p_i(jump from i at time t) = $q_i e^{-q_i t}$

<u>Jumping Chain</u>

$$J_{ij} = \begin{cases} 1 & j = 1 \quad q_i = 0 \\ q_{ij}/q_i & j \neq i \\ 0 & \end{cases} \Big\} q_i > 0$$

chain resides in i till a time T where T is distributed $p(T=t) = q_i e^{-q_i t}$ then jumps to j with pr. J_{ij}.

closedness, persistence are same for MP and Jumping C.

Yet positive / null, π_{ij} are not.

* <u>class structure, persis/trans, closed classes absorp prob.'s</u> are the same for the process, a τ skeleton, and J. chain

<u>One persistant Class Case:</u>

$xJ = x$ has a unique sol^n $\Rightarrow$ $x'Q = 0$ by $x'_n = x_n/q_n$

now $T_t(j|i)$ = time spent in state j up to time t | start at i

$T_t(j|i)/H_n(j|i) \longrightarrow 1/q_j$ by SLLN

$$\therefore \quad T_t(j|i)/T_t(k|i) = \left(\frac{H_n(j|i)}{H_n(k|i)}\right)\left(\frac{T_t(j|i)/H_n(j|i)}{T_t(k|i)/H_n(k|i)}\right) \xrightarrow{a.s.} \frac{x_j}{x_k}\cdot\frac{q_k}{q_j} = x'_j/x'_k .$$

Integral Equations

$p_{ij}(t, N) = pr_i(X(t) = j$ and at most N jumps by $t)$

$$p_{ij}(t\ N+1) = \delta_{ij} e^{-q_i t} + \int_0^t \sum_{\alpha \neq i} \frac{q_{i\alpha}}{q_i} p_{\alpha j}(u, N)\, q_i e^{-q_i(t-u)}\, du$$

let $N \to \infty$

$$p_{ij}(t) = \delta_{ij} e^{-q_i t} + \int_0^t \sum_{\alpha \neq i} \frac{q_{i\alpha}}{q_i} p_{\alpha j}(u) q_i\, e^{-q_i(t-u)}\, du \qquad \text{BIE}$$

gives

$$p_{ij}'(t) = \sum_{\alpha} q_{i\alpha} p_{\alpha j}(t) \qquad \dot{P} = QP$$

also

$$p_{ij}(t) = \delta_{ij} e^{-q_i t} + \int_0^t \sum_{\alpha \neq j} p_{i\alpha}(u) \left(\frac{q_{\alpha j}}{q_\alpha}\right) (q_\alpha e^{-q_j(t-u)})\, du \qquad \text{FIE.}$$

giving $\dot{P} = PQ$ need pf by induction

Positivity - Nullity

$p_{ii}(t) \longrightarrow$ limit >0 positive $\quad \lambda Q = 0$ null iff $\sum \lambda_i = \infty$

$= 0$ null

in the positive case $\pi_{ij} = \lim_{t \to \infty} p_{ij}(t) = \lambda_j / \sum \lambda_\alpha$

<u>ABC conditions:</u>

- $Qy = y$ has no bdd sol^n save $y=0 \Leftrightarrow$ pr(explosion) $= 0$

Let $\Delta_i(t) = 1 - \sum_j \Phi_{ij}(t) =$ pr$_i$(explosion)

$$1 - \Delta_i(t) = \sum_j \Phi_{ij}(t) = e^{-q_i t} + \sum_{\alpha \neq i} \int_0^t e^{-q_i u} q_{i\alpha} \{1 - \Delta_\alpha(t-u)\} du$$

$\Delta_i(t) = \sum_{\alpha\neq i} \int_0^t q_{i\alpha} e^{-q_i u} \Delta_\alpha(t-u) du$ Set $\Delta_i^*(t) = \int_0^\infty e^{-st} \Delta_i(s) ds$

$$\therefore \Delta_i^*(s) = \sum_{\alpha\neq i} \int_0^\infty \int_0^t q_{i\alpha} e^{-q_i u - st} \Delta_\alpha(t-u) du\, dt$$

$$= \sum_{\alpha\neq i} \int_0^\infty dt \int_0^t q_{i\alpha}\, e^{-q_i t - st + q_i v} \Delta_\alpha(v) dv$$

$$= \sum_{\alpha\neq i} \int_v^\infty dt \int_0^\infty q_{i\alpha}\, e^{-q_i t - st + q_i v} \Delta_\alpha(v) dv$$

$$= \sum_{\alpha\neq i} \frac{q_{i\alpha}}{q_i + s} \Delta_\alpha^*(s) \Rightarrow (q_i + s)\Delta_i^*(s) = \sum_{\alpha\neq i} \Delta_\alpha^*(s)$$

$\Rightarrow s\Delta_i^*(s) = (Q\Delta_i^*(s))$; if $y_i = \Delta_i^*(1)$ $Qy_i = y$

$\therefore \int_0^\infty e^{-t} \Delta_i(t) dt = 0 \ \forall i \Rightarrow \Delta_i(t) = 0 \Rightarrow$ pr$_i$(explosion) $= 0$.

Conversely: Let $\Delta_i^n(t) = 1 - \sum f_{ij}^n(t)$. $f_{ij}^n(t) =$ pr$_i$($\leq n$ jumps to j at t)

$\Delta_i^{(n)}(t) \longrightarrow \Delta_i(t)$

$\Delta_i^{n+1}(t) = \sum_{\alpha \neq i} \int_0^t e^{-q_i u} q_{i\alpha} \Delta_\alpha^n(t-u)\,du \qquad \Delta_i^{-1}(t) = 1$

$\Delta^{*\,n+1}_i(s) = \sum_{\alpha \neq i} \frac{q_{i\alpha}}{q_i + s} \Delta_i^{*n}(s) \qquad \Delta_i^{*-1}(s) = 1/s$

Suppose $Qy = y \qquad y_i = \sum_{\alpha \neq i} \frac{q_{i\alpha}}{q_i + 1} y_\alpha \leq \sum_{\alpha \neq i} \frac{q_{i\alpha}}{q_i + 1} \Delta_\alpha^{*-1}(1)$ say $= \Delta_i^{*0}(1)$

induct^n $\Rightarrow \quad y_i \leq \Delta_i^{*n}(1)$

$\Rightarrow \quad 0 \leq y_i \leq \Delta_i^*(1) = \int_0^\infty e^{-t} \Delta_i(t)\,dt \quad \Rightarrow \quad \Delta_i(t) > 0$ some i, t.