A theorem on minimal sufficient statistics

Recall that X has pmf/pdf $f_X(x;\theta)$ for $x \in \mathcal{X}$ and $\theta \in \Theta \subseteq \mathbb{R}^d$.

Theorem. *Suppose $T = t(X)$ is a statistic such that $f_X(x;\theta)/f_X(y;\theta)$ is constant as a function of θ if and only if $t(x) = t(y)$. Then T is minimal sufficient for θ.*

*Proof (**Non-examinable**).* We first show sufficiency. Define an equivalence relation $\sim$ on $\mathcal{X}$ by setting $x \sim y$ when $t(x) = t(y)$. (It is an easy exercise to check this is indeed an equivalence relation.) Let $\mathcal{U} := \{t(x) : x \in \mathcal{X}\}$ and for each $u \in \mathcal{U}$, choose a representative x_u from the equivalence class $\{x : t(x) = u\}$. Since $t(x) = t(x_{t(x)})$, we have by hypothesis that $f_X(x;\theta)/f_X(x_{t(x)};\theta)$ does not depend on θ, so we call this ratio $h(x)$. Setting $g(u,\theta) := f_X(x_u;\theta)$, we have

$$f_X(x;\theta) = f_X(x_{t(x)};\theta)\frac{f_X(x;\theta)}{f_X(x_{t(x)};\theta)} = g(t(x),\theta)h(x),$$

so by the Factorisation Criterion, $T = t(X)$ is sufficient.

To show minimal sufficiency, suppose $T' = t'(X)$ is another sufficient statistic with $t'(x) = t'(y)$. By the Factorisation Criterion, we can find functions g', h' such that $f_X(x;\theta) = g'(t'(x),\theta)h'(x)$. Thus

$$\frac{f_X(x;\theta)}{f_X(y;\theta)} = \frac{g'(t'(x),\theta)h'(x)}{g'(t'(y),\theta)h'(y)} = \frac{h'(x)}{h'(y)},$$

because $t'(x) = t'(y)$. Since this ratio therefore does not depend on θ, we conclude that $t(x) = t(y)$. Hence T is a function of T', so $T = t(X)$ is minimal sufficient. $\square$