1. Ask your supervisor to test you on the sheet of common distributions handed out in lectures.

2. (Probability review) If \( X \sim \text{Exp}(\lambda) \) and \( Y \sim \text{Exp}(\mu) \) are independent, derive the distribution of \( \min(X, Y) \). If \( X \sim \Gamma(\alpha, \lambda) \) and \( Y \sim \Gamma(\beta, \lambda) \) are independent, derive the distributions of \( X + Y \) and \( X/(X + Y) \).

3. Suppose that \( X \) and \( T \) have probability density functions \( f_X(x; \theta) \) and \( f_T(u; \theta) \) respectively. Prove the factorisation criterion for the sufficiency of \( T \).

4. (a) Let \( X_1, \ldots, X_n \) be independent Poisson random variables, with \( X_i \) having parameter \( i\theta \) for some \( \theta > 0 \). Find a real-valued sufficient statistic \( T \), and compute its distribution. Show that the maximum likelihood estimator \( \hat{\theta} \) of \( \theta \) is a function of \( T \), and show that it is unbiased.

(b) For some \( n > 2 \), let \( X_1, \ldots, X_n \sim \text{Exp}(\theta) \). Find a minimal sufficient statistic \( T \), and compute its distribution. Show that the maximum likelihood estimator \( \hat{\theta} \) of \( \theta \) is a function of \( T \), and is biased but asymptotically unbiased. Find an injective function \( h \) on \((0, \infty)\) such that, writing \( \psi = h(\theta) \), the maximum likelihood estimator \( \hat{\psi} \) of the new parameter \( \psi \) is unbiased.

5. For some \( n \geq 2 \) let \( X_1, \ldots, X_n \sim U[\theta, 2\theta] \), for some \( \theta > 0 \). Show that \( \tilde{\theta} = \frac{2}{3}X_1 \) is an unbiased estimator of \( \theta \). Use the Rao–Blackwell theorem to find an unbiased estimator \( \hat{\theta} \) which is a function of a minimal sufficient statistic and which satisfies \( \text{Var}_\theta(\hat{\theta}) < \text{Var}_\theta(\tilde{\theta}) \) for all \( \theta > 0 \).

6. Let \( X_1, \ldots, X_n \sim U[0, \theta] \). Find the maximum likelihood estimator \( \hat{\theta} \) of \( \theta \). By considering the distribution of \( \hat{\theta}/\theta \) and for \( \alpha \in (0, 1) \), find an appropriate, one-sided \( 100(1 - \alpha)\% \) confidence interval for \( \theta \) based on \( \hat{\theta} \).

7. Suppose that \( X_1 \sim N(\theta_1, 1) \) and \( X_2 \sim N(\theta_2, 1) \) independently, where \( \theta_1 \) and \( \theta_2 \) are unknown. Show that both the square \( S \) and circle \( C \) in \( \mathbb{R}^2 \), given by

\[
S = \{(\theta_1, \theta_2) : |\theta_1 - X_1| \leq 2.236, |\theta_2 - X_2| \leq 2.236\}
\]

\[
C = \{(\theta_1, \theta_2) : (\theta_1 - X_1)^2 + (\theta_2 - X_2)^2 \leq 5.991\}
\]

are 95\% confidence sets for \((\theta_1, \theta_2)\). Hint: \( \Phi(2.236) = (1 + \sqrt{0.95})/2 \), where \( \Phi \) is the distribution function of the \( N(0, 1) \) distribution. What might be a sensible criterion for choosing between \( S \) and \( C \)?
8. Suppose that the number of defects on a roll of magnetic recording tape can be modelled with Poisson distribution for which the parameter $\lambda$ is known to be either 1 or 1.5. Suppose the prior mass function for $\lambda$ is

$$\pi_\lambda(1) = 0.4, \quad \pi_\lambda(1.5) = 0.6.$$ 

A random sample of five rolls of tape finds $x = (3, 1, 4, 6, 2)$ defects respectively. Show that the posterior distribution for $\lambda$ given $x$ is

$$\pi_{\lambda|X}(1|x) = 0.012, \quad \pi_{\lambda|X}(1.5|x) = 0.988.$$ 

9. Let $X_1, \ldots, X_n$ be independent and identically distributed with conditional probability density function $f(x|\theta) = \theta x^{\theta-1}I_{\{x \in (0,1]\}}$ for some $\theta > 0$. Suppose the prior distribution for $\theta$ is $\Gamma(\alpha, \lambda)$. Find the posterior distribution of $\theta$ given $X = (X_1, \ldots, X_n)$ and the Bayesian point estimator of $\theta$ under the quadratic loss function.

10. (Law of small numbers) For each $n \in \mathbb{N}$, let $X_{n1}, \ldots, X_{nn} \overset{iid}{\sim} \text{Bernoulli}(p_n)$ and let $S_n = \sum_{i=1}^{n} X_{ni}$. Prove that if $np_n \to \lambda \in (0, \infty)$ as $n \to \infty$, then for each $x \in \{0, 1, \ldots\}$,

$$\mathbb{P}(S_n = x) \to \mathbb{P}(Y = x)$$ 

as $n \to \infty$, where $Y \sim \text{Poi}(\lambda)$.

11. For some $n \geq 3$, let $\epsilon_1, \ldots, \epsilon_n \overset{iid}{\sim} N(0, 1)$, set $X_1 = \epsilon_1$ and $X_i = \theta X_{i-1} + (1 - \theta^2)^{1/2} \epsilon_i$ for $i = 2, \ldots, n$ and some $\theta \in (-1, 1)$. Find a sufficient statistic for $\theta$ that takes values in a subset of $\mathbb{R}^3$.

12. Let $\hat{\theta}$ be an unbiased estimator of $\theta \in \Theta = \mathbb{R}$ satisfying $\mathbb{E}_\theta(\hat{\theta}^2) < \infty$ for all $\theta \in \Theta$. We say $\hat{\theta}$ is a uniform minimum variance unbiased (UMVU) estimator if $\text{Var}_\theta \hat{\theta} \leq \text{Var}_\theta \theta$ for all $\theta \in \Theta$ and any other unbiased estimator $\tilde{\theta}$. Prove that a necessary and sufficient condition for $\hat{\theta}$ to be a UMVU estimator is that $\mathbb{E}_\theta(\hat{\theta}^2) = 0$ for all $\theta \in \Theta$ and all estimators $U$ with $\mathbb{E}_\theta(U) = 0$ and $\mathbb{E}_\theta(U^2) < \infty$ (i.e. $\hat{\theta}$ is uncorrelated with every unbiased estimator of 0). Is the estimator $\hat{\theta}$ in 5. a UMVU estimator?