Proposition 12

In the following, let $C \subseteq \mathbb{R}^d$ be a convex set and let $f : C \to \mathbb{R}$ be a convex function, unless specified otherwise.

New convex functions from old:

(i) Let $g : C \to \mathbb{R}$ be a (strictly) convex function. Then if $a, b > 0$, $af + bg$ is a (strictly) convex function.

(ii) Let $A \in \mathbb{R}^{d \times m}$ and $b \in \mathbb{R}^d$ and take $C = \mathbb{R}^d$. Then $g : \mathbb{R}^m \to \mathbb{R}$ given by $g(x) = f(Ax - b)$ is a convex function.

(iii) Suppose $f_\alpha : C \to \mathbb{R}$ is convex for all $\alpha \in I$ where I is some index set, and define $g(x) := \sup_{\alpha \in I} f_\alpha(x)$. Then
   1. $D := \{x \in C : g(x) < \infty\}$ is convex and
   2. function g restricted to D is convex.

Proposition 12 continued

Consequences of convexity:

- iv For all $M \in \mathbb{R}$, the sublevel set $\{x \in C : f(x) \leq M\}$ is convex.
- v If f is differentiable at $x \in \text{int}(C)$ then $f(y) \geq f(x) + \nabla f(x)^\top (y - x)$ for all $y \in C$. In particular, $\nabla f(x) = 0 \Rightarrow x$ minimises f.
- vi If f is a strictly convex function, then any minimiser is unique.

Checking convexity:

- viii If $f : \mathbb{R}^d \to \mathbb{R}$ is twice continuously differentiable then
  1. f is convex iff. its Hessian matrix $H(x)$ at x is positive semi-definite for all x,
  2. f is strictly convex if $H(x)$ is positive definite for all x.

Convex surrogates