Stochastic Calculus and Applications (Lent 2020)

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The following references will be assumed:
[PM] J. Norris, Probability and Measure, `http://www.statslab.cam.ac.uk/~james/Lectures/pm.pdf`.
[AP] J. Norris, Advanced Probability, `http://www.statslab.cam.ac.uk/~james/Lectures/ap.pdf`.

1. Introduction

1.1. Motivation

ODE: $\mathring{x}(t) = F(x(t))$ — fundamental in analysis

SDE: $\mathring{x}(t) = F(x(t)) + \eta(t)$

random noise

What should η be?

For $|t-s| \gg 0$, $\eta(t)$ and $\eta(s)$ should be essentially independent.

→ Idealisation: $\eta(t)$ and $\eta(s)$ should be independent if $s \neq t$.

Such an η does not exist as a random function, but it exists as a random Schwartz distribution and is called White Noise.

If η was a function, for any $0 = t_0 < t_1 < \cdots$, the increments

$$x(t_i) - x(t_{i-1}) = \int_{t_{i-1}}^{t_i} \eta(s)\, ds$$

should be independent and their variance proportional to $|t_i - t_{i-1}|$ (by subdividing).

→ x should be Brownian motion and "$\mathring{B} = \eta$".

BM is not classically differentiable, but we can interpret the SDE as the integral equation

$$x(t) - x(0) = \int_0^t F(x(s)) + B_t.$$

This equation can be solved if F is Lipschitz, for example.

If g is a smooth function, does $g(x(t))$ also satisfy an equation? If $x(t)$ was a solution to an ODE, we would get

$$\frac{d}{dt} g(x(t)) = g'(x(t))\, \dot{x}(t).$$

Since η is not a function, this is turns out to be more subtle for SDEs.

One of the main goals of this course is to understand how to make sense of this.

1.2. The Wiener Integral

Defn. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. $S \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Gaussian space if S is a closed linear subspace and any $X \in S$ is a Gaussian random variable.

Example. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be any probability space on which there is a sequence of independent random variables $X_i \sim \mathcal{N}(0,1)$. Then (X_i) is an orthonormal system in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, i.e.,

$$\mathbb{E} X_i X_j = \delta_{ij}$$

and $S = \overline{\operatorname{span}\{X_i\}}$ is a Gaussian space. (Exercise: the limit in L^2 of a sequence of Gaussian random variables is again Gaussian.)

Prop. Let H be a separable Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P})$ as in the example. Then there is an isometry $I : H \longrightarrow S$. Thus

- For every $f \in H$, there is a random variable $I(f)$ where $I(f) \sim \mathcal{N}(0, (f,f)_H)$.
- For $f, g \in H$, one has $\mathbb{E} I(f) I(g) = (f,g)_H$.

In fact, $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ a.s.

Proof. Let $(e_i)_{i=1}^{\infty}$ be an orthonormal Hilbert basis for H. For $f \in H$, set

$$I(f) = \sum_{i=0}^{\infty} (f, e_i) X_i .$$

The limit exists in L^2 since $\sum_{i=1}^{k}(f,e_i)X_i$ is a Cauchy sequence:

$$\mathbb{E}\left[\left|\sum_{i=1}^{k}(f,e_i)X_i - \sum_{i=1}^{\ell}(f,e_i)X_i\right|^2\right] \le \sum_{i=k}^{\ell}(f,e_i)^2 \to 0 \text{ since } f\in H.$$

In fact, $k \mapsto \sum_{i=1}^{k}(f,e_i)X_i$ is a martingale, so the limit also exists almost surely.

To see that the map I is an isometry it suffices to note that it maps the orthonormal basis (e_i) to the orthonormal system (X_i).

<u>Defn.</u> A <u>Gaussian White Noise</u> on $\mathbb{R}_+$ is an isometry WN from $L^2(\mathbb{R}_+)$ into a Gaussian space. For $A\subset\mathbb{R}_+$ a Borel set, we write $WN(A) = WN(\mathbb{1}_A)$.

<u>Prop.</u> (i) For $A\subset\mathbb{R}_+$ Borel with $|A|<\infty$, $WN(A)\sim N(0,|A|)$.

(ii) For $A,B\subset\mathbb{R}_+$ Borel with $A\cap B=\emptyset$, $WN(A)$ and $WN(B)$ are independent.

(iii) If $A=\bigcup_{i=1}^{\infty}A_i$ for disjoint sets A_i as above, then

$$WN(A) = \sum_{i=1}^{\infty} WN(A_i) \quad \text{in } L^2 \text{ and a.s.} \qquad (*)$$

<u>Proof.</u> (i) holds since $(\mathbb{1}_A,\mathbb{1}_A)=|A|$; (ii) holds since $\mathbb{E}\,WN(A)\,WN(B)=0$ and uncorrelated jointly Gaussian random variables are independent; (iii) follows from the fact that $M_n=\sum_{i=1}^{n}WN(A_i)$ is a martingale bounded in L^2.

Upshot. WN looks like a random measure,

$$A \in \mathcal{B}(\mathbb{R}_+) \mapsto WN(\omega, A) \quad \text{for } \omega \in \Omega,$$

but it is not. The event $E \subset \Omega$ on which the 'countable additivity' (*) holds depends on the sets A_i.

For $t \geq 0$, define $B_t = WN([0,t])$.

Fact. For any $t_1, \dots, t_n$, the vector $(B_{t_i})_{i=1}^n$ is jointly Gaussian and $\mathbb{E} B_s B_t = s \wedge t$ for all $s,t \geq 0$. Moreover, $B_0 = 0$ a.s. and

$$B_t - B_s \text{ is independent of } \sigma(B_r, r<s) \text{ and } \sim \mathcal{N}(0, t-s)$$

for $t \geq s$.

Prop. ($\to$ AP). There is a modification of (B_t) s.t. $t \mapsto B_t$ is continuous almost surely.

Defn. This process is called Brownian motion.

Example. Let $f \in L^2(\mathbb{R}_+)$ be a step function $f = \sum_{i=1}^n f_i \mathbb{1}_{[t_i, t_{i+1}]}$, $t_i < t_{i+1}$.

$$\Rightarrow \quad WN(f) = \sum_{i=1}^n f_i (B_{t_{i+1}} - B_{t_i})$$

This motivates the notation

$$WN(f) = \int f(s) \, dB_s.$$

But since B is not of bounded variation, this integral is not defined in the classical (Lebesgue-Stieltjes) sense.

1.3. The Lebesgue – Stieltjes Integral

For an interval $I \subset \mathbb{R}$, we always use the Borel σ-algebra.

Defn. Let $T > 0$.

- A signed measure μ on $[0,T]$ is the difference of two mutually singular finite positive measures $\mu_\pm$ on $[0,T]$. The decomposition $\mu = \mu_+ - \mu_-$ is called the Hahn-Jordan decomposition of μ.
- The total variation of $\mu = \mu_+ - \mu_-$ is the positive measure $|\mu| = \mu_+ + \mu_-$.

Prop. (Hahn-Jordan). For any positive measures μ_1 and μ_2 on $[0,T]$ there is a signed measure μ s.t. $\mu = \mu_1 - \mu_2$.

Proof. Let $\nu = \mu_1 + \mu_2$. By the Radon-Nikodym Theorem, there are Borel functions $f_i \geq 0$ on $[0,T]$ s.t.

$$\mu_i(dt) = f(t)\, \nu(dt).$$

Let $f(t) = f_1(t) - f_2(t)$. Then

$$(\mu_1 - \mu_2)(dt) = f(t)\, \nu(dt) = \underbrace{f(t)^+ \nu(dt)}_{\mu_+(dt)} - \underbrace{f(t)^- \nu(dt)}_{\mu_-(dt)}$$

where $f(t)^+ = f(t) \wedge 0$ and $f(t)^- = -f(t) \wedge 0$ are the positive and negative parts of $f(t)$. This gives the decomposition into disjoint measures μ_+ and μ_-.

Defn. Let $T>0$.

- A function $a:[0,T]\to\mathbb{R}$ is <u>càdlàg</u> (write $a\in D([0,T])$) if $a(t_+)=a(t)$ for all t and $a(t_-)$ exists for all t. (Here: $a(t_\pm)=\lim_{s\to 0^\pm} a(t+s)$.)

- The <u>total variation</u> of a function $a:[0,T]\to\mathbb{R}$ is

$$V_a(0,T)=\sup\Big\{\sum_{i=1}^n |a(t_i)-a(t_{i-1})| : \underbrace{0\le t_0<t_1<\cdots<t_n\le T}_{\text{partition } (t_i)_{i=0}^n \text{ of } [0,T]}\Big\}.$$

- A function $a:[0,T]\to\mathbb{R}$ is of <u>bounded variation</u> (write $a\in BV([0,T])$) if $V_a(0,T)<\infty$.

Prop.

(i) Let μ be a signed measure on $[0,T]$. Then $a(t)=\mu([0,t])$ is càdlàg and of bounded variation with

$$|\mu((0,T])| = V_a(0,T).$$

half open interval

(ii) Let $a:[0,T]\to\mathbb{R}$ be càdlàg and of bounded variation. Then there is a signed measure μ s.t. $a(t)=\mu([0,t])$.

Fact ($\to$ PM). The map $\nu\mapsto f$, $f(t)=\nu([0,t])$ is a bijection from finite <u>positive</u> measures on $[0,T]$ to increasing right-continuous functions (which are càdlàg).

<u>Proof of proposition.</u> (i) Let $\mu = \mu_+ - \mu_-$ be the Hahn-Jordan decomposition of μ. Then

$$a(t) = \mu([0,t]) = \underbrace{\mu_+([0,t])}_{a_+(t)} - \underbrace{\mu_-([0,t])}_{a_-(t)} \quad \text{is càdlàg}$$

since $a_\pm$ are càdlàg.

Claim: $V_a(0,t) \leq |\mu|((0,t])$

Indeed, for any partition $0 \leq t_0 < t_1 < \cdots < t_n \leq t$ of $[0,t]$,

$$\sum_{i=1}^{n} |a(t_i) - a(t_{i-1})| = \sum_{i=1}^{n} |\mu((t_{i-1}, t_i])| \leq |\mu|((0,t])$$

$\Rightarrow V_a(0,t) \leq |\mu|((0,t])$.

Claim: For any nested sequence of partitions $(t_i^{(m)})_{i=0}^{n_m}$ with step size $\Delta(t^{(m)}) = \max_{i=1,\dots,n_m} |t_i^{(m)} - t_{i-1}^{(m)}| \to 0$ as $m \to \infty$,

$$|\mu|((0,t]) = \lim_{m \to \infty} \sum_{i=1}^{n_m} |a(t_i^{(m)}) - a(t_{i-1}^{(m)})|$$

Thus $|\mu|((0,t]) \leq V_a(0,t)$.

Indeed, consider the probability measure $\nu(dt) = \dfrac{|\mu|(dt)}{|\mu|((0,t])}$ on $(0,t]$. Let

$$\mathcal{F}_m = \sigma((t_{i-1}^{(m)}, t_i^{(m)}] : 1 \leq i \leq n_m) \subseteq \mathcal{F}_{m+1},$$

$$X = \frac{d\mu}{d|\mu|} = 1_{\operatorname{supp}\mu_+} - 1_{\operatorname{supp}\mu_-},$$

$$X_m = \mathbb{E}[X \mid \mathcal{F}_m].$$

For $s \in (t_{i-1}^{(m)}, t_i^{(m)}]$ then

$$X_m(s) = \frac{\mu((t_{i-1}^{(m)}, t_i^{(m)}])}{|\mu|((t_{i-1}^{(m)}, t_i^{(m)}])} = \frac{a(t_i^{(m)}) - a(t_{i-1}^{(m)})}{|\mu|((t_{i-1}^{(m)}, t_i^{(m)}])}$$

$$\Rightarrow \quad \mathbb{E}|X_m| = \frac{1}{|\mu|((0,t])} \sum_{i=1}^{n_m} |a(t_{i-1}^{(m)}) - a(t_i^{(m)})|$$

and the claim is $\mathbb{E}|X_m| \longrightarrow 1$. But indeed, (X_m) is a bounded martingale, so there is a random variable Y s.t. $X_n \to Y$ a.s. and in L^1. Thus

$$\mathbb{E}|X_m| \longrightarrow \mathbb{E}|Y|.$$

But since $\sigma(\bigcup_m \mathcal{F}_m) = \mathcal{B}((0,t])$ we have $X = Y$ a.s.

The claim follows from $\mathbb{E}|X| = 1$.

(ii) Let $a : [0,T] \to \mathbb{R}$ be as in the statement. We may assume that $a(0) = 0$. Set

$$a_\pm(t) = \frac{1}{2}\Big(V_a(0,t) \pm a(t)\Big).$$

Claim: $a_\pm$ are increasing

Let $s > t$. Let $0 \le t_0 \le \cdots \le t_n \le t$ be a partition of $[0,t]$ and note that $0 \le t_0 \le \cdots \le t_n \le t \le s$ is a partition of $[0,s]$.

$$\Rightarrow 2a_\pm(s) = V_a(0,s) \pm a(s) \ge \underbrace{\sum_{i=1}^{n} |a(t_{i-1}) - a(t_i)|}_{\ge V_a(0,t) - \varepsilon} + \underbrace{|a(s) - a(t)| \pm a(s)}_{\ge \pm a(t)}$$

$\Rightarrow a_\pm(s) \geq a_\pm(t) \Rightarrow a_\pm$ is increasing.

Claim ($\to$ Example Sheet). V_a is right-continuous

$\Rightarrow a_\pm$ is right-continuous

$\Rightarrow a_\pm(t) = \tilde{\mu}_\pm((0,t])$ for positive finite measures $\tilde{\mu}_\pm$.

Let $\mu = \tilde{\mu}_+ - \tilde{\mu}_-$. By the Hahn-Jordan Theorem, μ is a signed measure and

$$a(t) = a_+(t) - a_-(t) = \mu((0,t]).$$

<u>Example</u>. Define $a : [0,1] \to \mathbb{R}$ by

$$a(t) = \begin{cases} 1 & (t < \frac{1}{2}) \\ 0 & (t \geq \frac{1}{2}). \end{cases}$$

$\Rightarrow V_a(0,1) = 1, \ \mu = \delta_0 - \delta_{1/2}, \ |\mu| = \delta_0 + \delta_{1/2}.$

<u>Defn</u>. Let a be càdlàg and of bounded variation with associated signed measure μ. For $f \in L^1([0,T], |\mu|)$, the <u>Lebesgue-Stieltjes</u> integral is defined by

$$\int_s^t f(u)\, da(u) = \int_{(s,t]} f(u)\, \mu(du) \qquad (0 \leq s < t \leq T)$$

$$\int_s^t f(u)\, |da(u)| = \int_{(s,t]} f(u)\, |\mu|(du)$$

Also write $(f \circ a)(t) = \int_0^t f(u)\, da(u)$.

Fact. Let $a : [0,T] \to \mathbb{R}$ be càdlàg and BV. Then
$$\left| \int_s^t f(u)\, da(u) \right| \leq \int_s^t |f(u)|\, |da(u)|$$

and the function $f \circ a : [0,T] \to \mathbb{R}$ is càdlàg and BV with signed measure given by $f(u)\, da(u)$ and total variation $|f(u)|\, |da(u)|$.

Prop. Let a be càdlàg and BV. Let f be bounded and left-continuous. Let $(t_i^{(m)})$ be a sequence of partitions of $[0,t]$ with $\Delta(t^{(m)}) \longrightarrow 0$ as $m \to \infty$. Then
$$\int_0^t f(u)\, da(u) = \lim_{m\to\infty} \sum_{i=1}^{n_m} f(t_{i-1}^{(m)}) \left(a(t_i^{(m)}) - a(t_{i-1}^{(m)}) \right)$$
$$\int_0^t f(u)\, |da(u)| = \lim_{m\to\infty} \sum_{i=1}^{n_m} f(t_{i-1}^{(m)})\, \left| a(t_i^{(m)}) - a(t_{i-1}^{(m)}) \right|.$$

Proof. Define $f_m(0) = 0$, $f_m(s) = f(t_{i-1}^{(m)})$ if $s \in (t_{i-1}^{(m)}, t_i^{(m)}]$.

$\Rightarrow$ $f(s) = \lim_{m\to\infty} f_m(s)$ by left-continuity.

$\Rightarrow$ $\displaystyle\sum_{i=1}^{n_m} f(t_{i-1}^{(m)})(a(t_i^{(m)}) - a(t_{i-1}^{(m)})) = \int_{[0,t]} f_m(s)\, \mu(ds) \xrightarrow{\text{DCT}} \int_{[0,t]} f(s)\, \mu(ds)$

Similarly, $\int f_m \, d|\mu| \to \int f \, d|\mu|$, so for the second claim, it suffices to show $\left| \sum f(t_{i-1}^{(m)}) |a(t_i^{(m)}) - a(t_{i-1}^{(m)})| - \int f_m \, d|\mu| \right| \longrightarrow 0$ (Exercise).

Defn. A function $a : [0,\infty) \to \mathbb{R}$ is of <u>finite variation</u> ($a \in FV$) if $a|_{[0,T]} \in BV$ for all $T > 0$.

2. Semimartingales

From now on, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space.

Defn. For a process $X : \Omega \times [0,\infty) \to \mathbb{R}$ we say that

- X is adapted if $X_t = X(\cdot, t)$ is $\mathcal{F}_t$-measurable for all t;
- X is càdlàg if $X(\omega, \cdot) : [0,\infty) \to \mathbb{R}$ is càdlàg for all $\omega \in \Omega$;
- X is continuous, increasing, etc. analogously.

Notation: write $X \in \mathcal{F}$ to denote that a random variable X is measurable w.r.t. a σ-algebra $\mathcal{F}$.

2.1. Finite variation processes

Defn. (i) A finite variation process is a process that is càdlàg, adapted, and has finite variation for all $\omega \in \Omega$.

(ii) The total variation process associated to a finite variation process A is

$$V_t = \int_0^t |dA_t|.$$

Fact. The total variation process is càdlàg adapted and increasing.

Proof. That V is càdlàg adapted follows from deterministic properties of the Lebesgue–Stieltjes integral (prev. section).

To show that V is adapted, let $(t_i^{(m)})$ be a nested sequence of partitions of $[0,t]$ with $\Delta(t^{(m)}) \longrightarrow 0$. We have seen that then

$$V_t = \lim_{m\to\infty} \sum_{i=1}^{n_m} |A_{t_i^{(m)}} - A_{t_{i-1}^{(m)}}| \in \mathcal{F}_t$$

because measurability is preserved under pointwise limits. Thus V is adapted.

<u>Defn.</u> Let A be a <u>finite variation process</u> and let H be a process such that

$$\forall \omega\in\Omega \ \forall t>0 : \int_0^t |H_s(\omega)| \, |dA_s(\omega)| < \infty .$$

Then the process $H\cdot A = ((H\cdot A)_t)_{t\geq 0}$ is defined by

$$(H\cdot A)_t = \int_0^t H_s \, dA_s .$$

To show that $H\cdot A$ is adapted, we need a condition.

<u>Defn.</u> The <u>predictable σ-algebra</u> $\mathcal{P}$ on $\Omega\times[0,\infty)$ is the σ-algebra generated by the sets

$$E\times(s,t], \quad E\in\mathcal{F}_s, \ s<t.$$

A process $H : \Omega\times[0,\infty)\to\mathbb{R}$ is <u>predictable</u> if it is $\mathcal{P}$-measurable.

Defn. A process H is simple (write $H \in \mathcal{E}$) if
$$H(\omega,t) = \sum_{i=1}^{n} H_{i-1}(\omega) \mathbb{1}_{(t_{i-1},t_i]}(t)$$
for bounded random variables $H_i \in \mathcal{F}_{t_i}$ and $0=t_0<t_1<\cdots$

Fact. Simple processes and their pointwise limits are predictable. Hence adapted left-continuous processes are predictable.

Proof. The first claim is clear from the definition and the fact that measurability is preserved under pointwise limits. Let H be adapted left-continuous. Then $H_t^n \to H_t$ where
$$H_t^n = \sum_{i=1}^{n} H_{(i-1)2^{-n}} \mathbb{1}_{((i-1)2^{-n}, i2^{-n}]}(t) \wedge n.$$
Since H is adapted, H^n is simple and thus predictable. Since H is a pointwise limit, it is also predictable.

Fact. Let H be predictable. Then $H_t \in \mathcal{F}_{t-} := \sigma(\mathcal{F}_s : s<t)$.

Fact. Let H be adapted càdlàg. Then X_{t-} is adapted left-continuous, so predictable.

Example. • BM is predictable since continuous.

• A Poisson process (N_t) is not predictable since $N_t \notin \mathcal{F}_{t-}$.

Prop. Let A be a finite variation process, and let H be a process such that $\int_0^t |H_s(\omega)|\,|dA_s(\omega)| < \infty$ for all t, ω. Then $H \cdot A$ is also a finite variation process.

Proof. By properties given in Section 1.3, for every $\omega \in \Omega$, $(H\cdot A)(\omega, \cdot)$ is of finite variation and càdlàg. Thus it only remains to show that $H\cdot A$ is adapted.

First, this is true if $H(\omega,t) = 1_{(u,v]}(t)\, 1_E(\omega)$, $u<v$, $E \in \mathcal{F}_u$:

$$(H\cdot A)(\omega,t) = 1_E(\omega)\big(A(\omega, t\wedge v) - A(\omega, t\wedge u)\big)$$

$\Rightarrow (H\cdot A)_t \in \mathcal{F}_t$ since A is adapted

The general case follows from a monotone class argument. Let

$$\Pi = \{E \times (u,v] : u,v,\ E \in \mathcal{F}_u\} \subset \Omega \times [0,\infty).$$

Clearly, Π is a π-system (closed under intersection, nonempty) generating the predictable σ-algebra. Let

$$\mathcal{D} = \{H : \Omega \times [0,\infty) \to \mathbb{R} : H\cdot A \text{ is adapted}\}.$$

Then $1 \in \mathcal{D}$, $1_H \in \mathcal{D}$ for $H \in \Pi$, and if $0 \le H_n \in \mathcal{D}$ with $H_n \uparrow H$ then $H \in \mathcal{D}$ since measurability is preserved under pointwise limits. This means that $\mathcal{D}$ is a monotone class. By the monotone class theorem ($\to$ PM), $\mathcal{D}$ contains all bounded predictable processes.

For H unbounded, one can choose $H^n = (H \wedge n) \vee (-n)$ so that

$|H^n| \le |H|$ and $H^n \to H$ pointwise. By dominated convergence, $H_n \cdot A \to H \cdot A$ pointwise in ω, t. The claim is thus true since measurability is preserved under pointwise limits.

2.2. Local martingales

From now on, assume $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfies the usual conditions ($\to$ AP), i.e.,

- $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets;
- $(\mathcal{F}_t)$ is right-continuous, i.e., $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ for all s.

Thm. (OST) Let X be a càdlàg adapted integrable process. The following are equivalent:

(i) X is a martingale: $\mathbb{E}(X_t | \mathcal{F}_s) = X_s$ a.s. for all $t \ge s$;

(ii) for all stopping times S, T with T bounded,

$$X_T \in L^1 \text{ and } \mathbb{E}(X_T | \mathcal{F}_S) = X_{S \wedge T} \text{ a.s.}$$

(iii) for all stopping times T, the process X^T where $X^T_t = X_{t \wedge T}$ is a martingale.

(iv) for all bounded stopping times T,

$$X_T \in L^1 \text{ and } \mathbb{E}(X_T) = \mathbb{E}(X_0).$$

Moreover, if X is uniformly integrable, then (ii) & (iv) hold for all stopping times T.

<u>Defn</u> A càdlàg adapted process X is a <u>local martingale</u> if there are stopping times T_n such that $T_n \uparrow \infty$ as $n \to \infty$ and X^{T_n} is a martingale for every n. The sequence (T_n) is said to <u>reduce</u> X.

<u>Example.</u> (i) Every martingale is a local martingale. (Take $T_n = n$ and use OST.)

(ii) Let (B_t) be a standard BM on $\mathbb{R}^3$. Then $(X_t)_{t\geq 1} = (1/|B_t|)_{t\geq 1}$ is a local martingale but not a martingale.

<u>Proof.</u> First, (X_t) cannot be a martingale since ($\to$ AP)

$$\sup_{t\geq 1} \mathbb{E}|X_t|^2 < \infty, \quad \mathbb{E}X_t \to 0.$$

To see that (X_t) is a local martingale nonetheless, recall that for $f \in C_b^2(\mathbb{R}^3)$,

$$f(B_t) - f(B_1) - \frac{1}{2}\int_1^t \Delta f(B_s)\, ds =: M_t^f$$

is a martingale. We cannot choose $f(x) = 1/|x|$ since this f is not bounded near 0.

Let $f_n \in C_b^2(\mathbb{R}^3)$ be such that $f(x) = 1/|x|$ for $|x| > \frac{1}{n}$. Also let $T_n = \inf\{t \geq 1 : |B_t| < \frac{1}{n}\}$. Then

$$X_{t\wedge T_n} - X_1 = f_n(B_{t\wedge T_n}) - f_n(B_1) = M^{f_n}_{t\wedge T_n}$$

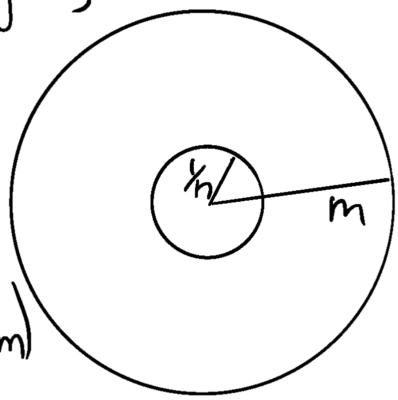
since $\Delta f_n = \Delta \frac{1}{|x|} = 0$ for $|x| \geq \frac{1}{n}$

which means that X^{T_n} is a martingale. To show that X is a local martingale, it remains to show that $T_n \to \infty$ a.s. The argument for this is a standard trick. Let

$$S_m = \inf\{t \geq 1 : |B_t| > m\}.$$

By OST, since X^{T_n} is a bounded martingale,

$$\mathbb{E} X_{T_n \wedge S_m} = \mathbb{E}X_1 < \infty.$$



But also

$$\mathbb{E} X_{T_n\wedge S_m} \geq n\,\mathbb{P}(T_n < S_m) + 0\,\mathbb{P}(T_n \geq S_m)$$

$$\Rightarrow \mathbb{P}(T_n < \infty) = \mathbb{P}(T_n < \lim_{m\to\infty} S_m)$$

$$\leq \lim_{m\to\infty} \mathbb{P}(T_n < S_m) \leq \frac{1}{n}\mathbb{E}X_1$$

$$\Rightarrow \mathbb{P}(\lim_{n\to\infty} T_n < \infty) \leq \lim_{n\to\infty} \frac{1}{n} \mathbb{E}X_1 = 0, \text{ i.e. } T_n \to \infty \text{ a.s.}$$

Though $X = 1/|B_t|$ on $\mathbb{R}^3$ is not a martingale, the next proposition implies it is a supermartingale.

Prop. Let X be a nonnegative local martingale. Then X is a supermartingale.

Proof. Let (T_n) be a reducing sequence. Then

$$\mathbb{E}(X_t \mid \mathcal{F}_s) = \mathbb{E}\big(\lim_{n\to\infty} X_{t\wedge T_n} \mid \mathcal{F}_s\big)$$

$$\le \liminf_{n\to\infty} \mathbb{E}(X_{t\wedge T_n} \mid \mathcal{F}_s) \quad \text{by conditional Fatou}$$

$$= \liminf_{n\to\infty} X_{s\wedge T_n} = X_s \qquad \text{a.s.}$$

Prop. Let X be a local martingale and suppose that there is $Z \in L^1$ s.t. $|X_t| \le Z$ for all t. Then X is a martingale. In particular, bounded local martingales are martingales.

Proof. Exercise: prove directly (use DCT). It also follows from the following general criterion.

Lemma ($\to$ Example Sheet). Let $X \in L^1$. Then the set

$$\mathcal{X} = \{\, \mathbb{E}(X \mid \mathcal{G}) : \mathcal{G} \subset \mathcal{F} \text{ is a sub-}\sigma\text{-algebra} \,\}$$

is uniformly integrable (UI), i.e.,

$$\sup_{Y\in\mathcal{X}} \mathbb{E}(|Y| \mathbb{1}_{|Y|>\lambda}) \longrightarrow 0 \text{ as } \lambda \to \infty.$$

Vitali's Theorem. $X_n \to X$ in L^1 iff (X_n) is UI and $X_n \to X$ in probability.

Prop. The following are equivalent:

(a) X is a martingale.

(b) X is a local martingale and for all $t \geq 0$ the set $\mathcal{X}_t = \{X_T : T \text{ is a stopping time with } T \leq t\}$ is UI.

Proof. (a) $\Rightarrow$ (b). Let X be a martingale. By OST,

$$X_T = \mathbb{E}(X_t \mid \mathcal{F}_T) \text{ for any stopping time } T \leq t.$$

By the last lemma, thus $\mathcal{X}_t$ is UI.

(b) $\Rightarrow$ (a) Let X be a local martingale and assume $\mathcal{X}_t$ is UI for all t. By OST, it suffices to show that $\mathbb{E}X_T = \mathbb{E}X_0$ for any bounded stopping time T. To see this, let (T_n) be a reducing sequence for X. Then

$$\mathbb{E}X_0 = \mathbb{E}X_0^{T_n} = \mathbb{E}X_T^{T_n} = \mathbb{E}X_{T_n \wedge T}.$$

Since $T_n \wedge T \to T$ a.s. and $\{X_{T \wedge T_n} : n \geq 0\}$ is UI, it follows that $X_{T \wedge T_n} \to X_T$ in L^1. Thus $\mathbb{E}X_T = \mathbb{E}X_0$ and X is a martingale.

Fact. Let X be a continuous adapted process with $X_0 = 0$. Then $S_n = \inf\{t \geq 0 : |X_t| = n\}$ are stopping times and $S_n \uparrow \infty$ a.s. as $n \to \infty$.

<u>Prop.</u> Let X be a continuous local martingale. Then the sequence of the last fact reduces X.

<u>Proof.</u> Let (T_k) be a reducing sequence for X. By OST, $X^{T_k \wedge S_n}$ is a martingale, so X^{S_n} is also a local martingale. Since $|X^{S_n}| \leq n$, X^{S_n} is also bounded, so a true martingale. Thus (S_n) satisfies all conditions to be a reducing sequence for X.

<u>Thm.</u> Let X be a continuous local martingale with $X_0 = 0$. If X is also a finite variation process then $X_t = 0$ $\forall t$ a.s.

<u>Proof.</u> Let

$$S_n = \inf\{t \geq 0 : \int_0^t |dX_s| = n\}.$$

Since S_n is a stopping time, X^{S_n} is a local martingale by OST. It is also bounded since

$$|X_t^{S_n}| = \left|\int_0^{t\wedge S_n} dX_s\right| \leq \int_0^{t \wedge S_n} |dX_s| \leq n.$$

Thus X^{S_n} is a martingale. Let $(t_i)_{i=0}^k$ be a partition of $[0,t]$. Then

$$\mathbb{E}|X_t^{S_n}|^2 = \sum_{i=1}^k \mathbb{E}|X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}|^2 \quad \text{since } X^{S_n} \text{ is a martingale}$$

$$\leq \mathbb{E}\Bigg(\underbrace{\max_i |X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}|}_{\substack{\leq n \text{ and} \\ \to 0 \text{ as } \Delta(t) \to 0}} \underbrace{\sum_{i=1}^k |X_{t_i}^{S_n} - X_{t_{i-1}}^{S_n}|}_{\leq \int_0^{t\wedge S_n} |dX_s| \leq n}\Bigg)$$

Take $\Delta(t) \to 0$. By continuity and DCT,

$$\mathbb{E}|X_t^{S_n}|^2 = 0.$$

$\Rightarrow X_t^{S_n} = 0$ a.s. $\forall t \Rightarrow X_t = 0$ a.s. $\forall t$

$\Rightarrow X_t = 0 \ \forall t \in [0,\infty) \cap \mathbb{Q}$ a.s.

Using that X is continuous, hence $X_t = 0 \ \forall t$ a.s.

2.3. L^2 bounded martingales

Below $\sim$ means identification of indistinguishable processes.

Defn. Define the spaces

$$\mathcal{M}^2 = \{X : \Omega \times [0,\infty) \to \mathbb{R} : X \text{ is a càdlàg martingale with } \sup_{t \geq 0} \mathbb{E}|X_t|^2 < \infty\} / \sim$$

$$\mathcal{M}_c^2 = \{X \in \mathcal{M}^2 : X(\omega, \cdot) \text{ is continuous for all } \omega\} / \sim$$

with norm

$$\|X\|_{\mathcal{M}^2} = \left(\sup_{t \geq 0} \mathbb{E}X_t^2\right)^{1/2} = \left(\mathbb{E}X_\infty^2\right)^{1/2}.$$

Here recall that ($\to$ AP): if $X \in \mathcal{M}^2$ then

- $X_t \to X_\infty$ a.s. and in L^2
- (X_t^2) is a submartingale, so $t \mapsto \mathbb{E}X_t^2$ is increasing
- Doob's L^2 inequality: $\mathbb{E}(\sup_t X_t^2) \leq 4\mathbb{E}X_\infty^2$.

Prop. $\mathcal{M}^2$ is a Hilbert space with norm $\|\cdot\|_{\mathcal{M}^2}$ and inner product $\mathbb{E} X_\infty Y_\infty$. $\mathcal{M}^2_c$ is a closed subspace.

Proof. The only nontrivial property to prove is that $\mathcal{M}^2$ is complete. Thus let $(X^n) \subset \mathcal{M}^2$ be a Cauchy sequence:

$$\mathbb{E}(X^n_\infty - X^m_\infty)^2 \longrightarrow 0 \quad \text{as } n, m \to \infty.$$

By passing to a subsequence, we may assume that

$$\mathbb{E}(X^n_\infty - X^{n-1}_\infty)^2 \le 2^{-n}$$

and it suffices to show that the subsequence converges to conclude that the original sequence converges. Now

$$\mathbb{E}\Big(\sum_{n=1}^\infty \sup_{t\ge 0} |X^n_t - X^{n-1}_t|\Big) \overset{(CS)}{\le} \sum_{n=1}^\infty \mathbb{E}\Big(\sup_{t\ge 0} |X^n_t - X^{n-1}_t|^2\Big)^{1/2}$$

$$\overset{(Doob)}{\le} \sum_{n=1}^\infty 2\big(\mathbb{E}|X^n_\infty - X^{n-1}_\infty|^2\big)^{1/2} \le \sum_{n=1}^\infty 2^{1-n/2} < \infty.$$

$$\Rightarrow \sum_{n=1}^\infty \sup_{t\ge 0} |X^n_t - X^{n-1}_t| < \infty \quad \text{a.s.}$$

$\Rightarrow$ (X^n) is a Cauchy sequence in $D([0,\infty)), \|\cdot\|_\infty$ a.s.

By completeness of $D([0,\infty))$, therefore

$$\|X^n - X\|_\infty \to 0 \quad \text{for some } X \in D([0,\infty)) \quad \text{a.s.}$$

Set $X = 0$ outside the a.s. event. Then $X \in D([0,\infty))$ for all ω.

Claim: $\mathbb{E}\big(\sup_{t\ge 0} |X^n_t - X_t|^2\big) \to 0$.

Indeed,

$$\mathbb{E}\left(\sup_t |X^n_t - X_t|^2\right) = \mathbb{E}\left(\lim_{n\to\infty} \sup_t |X^n - X^m|^2\right)$$

$$\overset{\text{(Fatou)}}{\leq} \liminf_{n\to\infty} \mathbb{E}\left(\sup_t |X^n - X^m|^2\right)$$

$$\overset{\text{(Doob)}}{\leq} \liminf_{n\to\infty} 4\,\mathbb{E}\left(|X^n_\infty - X^m_\infty|^2\right) \to 0.$$

Claim: X is a martingale.

$$\| \mathbb{E}(X_t | \mathcal{F}_s) - X_s \|_{L^2} \leq \| \mathbb{E}(X_t - X^n_t | \mathcal{F}_s) \|_{L^2} + \| X^n_s - X_s \|_{L^2}$$

(Δ-inequality and X^n is a martingale)

$$\overset{\text{(Jensen)}}{\leq} \|X_t - X^n_t\|_{L^2} + \|X^n_s - X_s\|_{L^2}$$

$$\leq 2\,\mathbb{E}\left(\sup_t |X_t - X^n_t|\right) \to 0$$

Thus $X \in \mathcal{M}^2$ and we have shown that $\mathcal{M}^2$ is complete.

Clearly, $\mathcal{M}^2_c$ is a subspace of $\mathcal{M}^2$. It is complete (thus closed) by the same argument with $D([0,\infty))$ replaced by $C([0,\infty))$.

2.4. Quadratic variation

Defn. For a sequence of processes (X^n) and a process X,

$X^n \longrightarrow X$ uniformly on compacts in probability (UCP)

means that $\forall t \geq 0\ \forall \varepsilon \geq 0$: $\mathbb{P}\left(\sup_{0\leq s\leq t} |X^n_s - X_s| > \varepsilon\right) \longrightarrow 0$.

Thm. Let M be a continuous local martingale. Then there exists a unique (up to indist.) increasing process $\langle M\rangle = (\langle M\rangle_t)_{t\geq 0}$ s.t. $\langle M\rangle_0 = 0$ and $M^2 - \langle M\rangle$ is a continuous local martingale. Moreover, for any sequence of partitions $(t_i^{(m)})$ of $\mathbb{R}_+$ with $\Delta(t^{(m)}) \to 0$,

$$\langle M\rangle_t^{(m)} \to \langle M\rangle_t \text{ UCP, where } \langle M\rangle_t^{(m)} = \sum_{i=1}^{m} (M_{t_i\wedge t} - M_{t_{i-1}\wedge t})^2.$$

(We will only need and prove this for dyadic partitions.)

Defn. The process $\langle M\rangle$ is the quadratic variation of M.

Example. Let B be a standard Brownian motion. Then $B_t^2 - t$ is a martingale. Thus $\langle B\rangle_t = t$.

To prove the theorem, we may assume that $M_0 = 0$.

Lemma (uniqueness). There is a most one process $\langle M\rangle$ (up to indistinguishability) as asserted in the theorem.

Proof. Suppose both (A_t) and (B_t) obey the conditions asserted for $\langle M\rangle$. Then

$$\underbrace{A_t - B_t}_{\text{FV}} = \underbrace{(M_t^2 - B_t) - (M_t^2 - A_t)}_{\text{continuous local martingale}}$$

$\Rightarrow A = B$ a.s.

<u>Lemma</u> (stopping). Suppose M is a continuous local martingale for which $\langle M\rangle$ exists (as in the theorem). Let T be a stopping time. Then $\langle M^T\rangle$ exists and

$$\langle M^T\rangle_t = \langle M\rangle_{T\wedge t} \quad \text{(up to indistinguishability).}$$

<u>Proof.</u> Since $M_t^2 - \langle M\rangle_t$ is a continuous local martingale, so is $M_{t\wedge T}^2 - \langle M\rangle_{t\wedge T}$. The claim follows from uniqueness.

<u>Lemma</u> (finite case). Assume there are deterministic constants C and T such that

$$|M_t| \le C, \quad M_t = M_{t\wedge T}. \qquad (B)$$

Let $P = (t_i)_{i=1}^n$ be a partition of $[0,T]$ and set

$$\langle M\rangle_t^{(P)} = \sum_{i=1}^n \left(M_{t_i\wedge t} - M_{t_{i-1}\wedge t}\right)^2$$

$$X_t^{(P)} = \sum_{i=1}^n M_{t_{i-1}}\left(M_{t_i\wedge t} - M_{t_{i-1}\wedge t}\right).$$

Then: (i) $X^{(P)}$ is a bounded martingale

(ii) $\langle M\rangle_{t_k}^{(P)}$ is increasing in k

(iii) $M_{t_k}^2 - \langle M\rangle_{t_k}^{(P)} = 2X_{t_k}^{(P)}$ for $k=0,\dots,n$

(iv) $\mathbb{E}\left(\left(\langle M\rangle_\infty^{(P)}\right)^2\right) \le 12\, C^4$

Proof. (i) & (ii) obvious

(iii) $\langle M\rangle^{(P)}_{t_k} = \sum_{i=1}^{k} \underbrace{(M_{t_i} - M_{t_{i-1}})^2}$

$$\left[\begin{array}{l} M_{t_i}(M_{t_i} - M_{t_{i-1}}) - M_{t_{i-1}}(M_{t_i} - M_{t_{i-1}}) \\ = -2M_{t_{i-1}}(M_{t_i} - M_{t_{i-1}}) + (M_{t_i} + M_{t_{i-1}})(M_{t_i} - M_{t_{i-1}}) \end{array}\right.$$

$$= \sum_{i=1}^{k} (M_{t_i}^2 - M_{t_{i-1}}^2) - 2X^{(P)}_{t_k} = M_{t_k}^2 - 2X^{(P)}_{t_k}$$

(iv) $\mathbb{E}\left(\left(\langle M\rangle^{(P)}_\infty\right)^2\right) = \mathbb{E}\left(\left(\sum_{i=1}^{n}(M_{t_i} - M_{t_{i-1}})^2\right)^2\right)$

$$= \sum_{i=1}^{n} \mathbb{E}(M_{t_i} - M_{t_{i-1}})^4$$

$$+ 2\sum_{i=1}^{n} \underbrace{\mathbb{E}\left((M_{t_i} - M_{t_{i-1}})^2 \sum_{k=i+1}^{n} (M_{t_k} - M_{t_{k-1}})^2\right)}_{\mathbb{E}\left((M_{t_i} - M_{t_{i-1}})^2 (M_{t_n} - M_{t_i})^2\right) \text{ by orthogon.}}$$

$$\le (4+8)C^2 \sum_{i=1}^{n} \mathbb{E}(M_{t_i} - M_{t_{i-1}})^2$$

$$= 12C^2\, \mathbb{E}(M_{t_n} - M_{t_0})^2 \le 12C^4.$$

Lemma. Assume M satisfies (B). Let P^m be a sequence of nested partitions with $\Delta(P^m) \to 0$. Then $(X^{(P^m)})_m$ converges in $\mathcal{M}_c^2$ as $m \to \infty$.

For notational convenience, we will in fact assume that the partitions are dyadic:

$t_i = i2^{-m}$ where $i = 0, \dots, n_m = \lfloor 2^m T \rfloor$.

Proof. $X^m \in \mathcal{M}_c^2$ since M is a bounded martingale. For $m' > m$, one has

$$X_\infty^{m'} - X_\infty^m = \sum_{i=1}^{n_{m'}} \left(M_{(i-1)2^{-m'}} - M_{\lfloor (i-1)2^{m-m'} \rfloor 2^{-m}}\right)\left(M_{i2^{-m'}} - M_{(i-1)2^{-m'}}\right)$$

$$\Rightarrow \mathbb{E}(X_\infty^{m'} - X_\infty^m)^2 = \sum_{i=1}^{n_{m'}} \mathbb{E}\left(M_{(i-1)2^{-m'}} - M_{\lfloor (i-1)2^{m-m'} \rfloor 2^{-m}}\right)^2\left(M_{i2^{-m'}} - M_{(i-1)2^{-m'}}\right)^2$$

(orthogonal increments)

$$\leq \mathbb{E}\left(\sup_{|t-s| \leq 2^{-m}} |M_t - M_s|^2 \underbrace{\sum_{i=1}^{n_{m'}} \left(M_{i2^{-m'}} - M_{(i-1)2^{-m'}}\right)^2}_{\langle M \rangle_\infty^{(m')}}\right)$$

$$\leq \mathbb{E}\left(\sup_{|s-t| \leq 2^{-m}} |M_t - M_s|^4\right)^{1/2} \mathbb{E}\left(\left(\langle M \rangle_\infty^{(m')}\right)^2\right)^{1/2}$$

(Cauchy-Schwarz)

$$\leq (12C^2)^{1/2}\, \mathbb{E}\left(\sup_{|s-t| \leq 2^{-m}} |M_t - M_s|^4\right)^{1/2}$$

Since $|M_t - M_s| \leq (2C)^4$

$\sup_{|t-s| < 2^{-m}} |M_t - M_s| \longrightarrow 0$ by uniform continuity (bounded interval).

By DCT, thus $\mathbb{E}\left(\sup_{|t-s| \leq 2^{-m}} |M_t - M_s|^4\right) \longrightarrow 0$.

$\Rightarrow \mathbb{E}(X_\infty^{m'} - X_\infty^m)^2 \to 0$ as $m, m' \to \infty$: (X^m) is Cauchy in $\mathcal{M}_c^2$.

$\Rightarrow$ There is $X \in \mathcal{M}_c^2$ s.t. $X^m \to X$ in $\mathcal{M}_c^2$.

Let X be the limit from the previous lemma. Set

$$\langle M\rangle_t = M_t^2 - 2X_t.$$

Then $\langle M\rangle$ is continuous since M and X are.

$M^2 - \langle M\rangle = 2X$ is a martingale since X is.

$\langle M\rangle$ is increasing since $\langle M\rangle^{(P^m)}$ is increasing on the set of times $(t_k^{(m)}) = P^m$ and the convergence $M^2 - 2X^{(P^m)} \to \langle M\rangle$ is uniform in t by the next lemma.

<u>Lemma.</u> Assume (B). Then $\langle M\rangle^{(P^m)} \longrightarrow \langle M\rangle$ UCP.

<u>Proof</u> Since $\langle M\rangle_t^{(m)} = \langle M\rangle^{(m)}_{\lfloor 2^m t\rfloor 2^{-m}} + \left(M_t - M_{\lfloor 2^m t\rfloor 2^{-m}}\right)^2$

$$= M_t^2 - 2X^{(m)}_{\lfloor 2^m t\rfloor 2^{-m}} + 2\left(M^2_{\lfloor 2^m t\rfloor 2^{-m}} - M_t M_{\lfloor 2^m t\rfloor 2^{-m}}\right)$$

One has

$$\begin{aligned}\sup_t |\langle M\rangle_t - \langle M\rangle_t^{(m)}| \le{}& \sup_t 2\,|M_t M_{\lfloor 2^m t\rfloor 2^{-m}} - M^2_{\lfloor 2^m t\rfloor 2^{-m}}| \quad (=(\mathrm{I}))\\ &+ \sup_t 2\,|X_t - X_{\lfloor 2^m t\rfloor 2^{-m}}| \quad (=(\mathrm{II}))\\ &+ \sup_t 2\,|X_{\lfloor 2^m t\rfloor 2^{-m}} - X^{(m)}_{\lfloor 2^m t\rfloor 2^{-m}}| \quad (=(\mathrm{III}))\end{aligned}$$

(I) and (II) tend to 0 a.s. since M and X are uniformly continuous on $[0,T]$.

(III) tends to 0 in L^2 since $X^m \to X$ in $\mathcal{M}^2$ implies

$$\|\sup_t |X_t^n - X_t|\,\|_{L^2} \longrightarrow 0$$

by Doob's inequality.

Proof of theorem. Let M be a continuous local martingale. Let
$$T_n = \inf\{t \geq 0 : |M_t| > n\} \wedge n.$$
Then M^{T_n} satisfies (B), so $\langle M^{T_n}\rangle$ exists as above. By uniqueness,
$$\langle M^{T_n}\rangle_t = \langle M^{T_{n+1}}\rangle_{t\wedge T_n}.$$
Thus there is a process $\langle M\rangle$ s.t. $\langle M\rangle_{t\wedge T_n}$ and $\langle M^{T_n}\rangle_t$ are indistinguishable for all $n \in \mathbb{N}$.

$\langle M\rangle$ is increasing since the $\langle M^{T_n}\rangle$ are.

$M^2 - \langle M\rangle$ is a local martingale since $T_n \uparrow \infty$ a.s. and $(M^2 - \langle M\rangle)^{T_n} = (M^{T_n})^2 - \langle M^{T_n}\rangle$ is a martingale.

It remains to show that $\langle M\rangle^{(m)} \longrightarrow \langle M\rangle$ UCP.
$$\mathbb{P}\Big(\sup_{t\leq T} |\langle M\rangle_t^{(m)} - \langle M\rangle_t| > \varepsilon\Big) \leq \underbrace{\mathbb{P}(T_n < T)}_{\to 0} + \underbrace{\mathbb{P}\Big(\sup_{t\leq T}|\langle M^{S_n}\rangle_t^{(m)} - \langle M^{S_n}\rangle_t| > \varepsilon\Big)}_{\to 0 \text{ by last lemma}}$$

This completes the proof.

Prop Let M be a continuous local martingale with $M_0=0$. Then $M=0$ iff $\langle M\rangle=0$.

Proof. Clearly, $M=0$ implies $\langle M\rangle=0$. Conversely, if $\langle M\rangle=0$, then M^2 is a nonnegative local martingale, so also a supermartingale. Thus $\mathbb{E} M_t^2 \le \mathbb{E} M_0^2 = 0$ for all t. The claim now follows from continuity.

Prop Let M be a continuous local martingale with $M_0=0$. Then $M\in\mathcal{M}^2$ iff $\mathbb{E}\langle M\rangle_\infty<\infty$ and then $M^2-\langle M\rangle$ is a UI martingale and

$$\|M\|_{\mathcal{M}^2} = \left(\mathbb{E}\langle M\rangle_\infty\right)^{1/2}.$$

Proof First assume $M\in\mathcal{M}^2$ and $\mathbb{E}\langle M\rangle_\infty<\infty$. Then

$$|M_t^2-\langle M\rangle_t| \le \underbrace{\sup_{t\ge 0} M_t^2 + \langle M\rangle_\infty}_{=: Z\in L^1 \text{ by Doob's inequality}}.$$

Thus $M^2-\langle M\rangle$ is UI and we have also seen that $M^2-\langle M\rangle$ is a (true) martingale starting at 0. Therefore

$$\|M\|_{\mathcal{M}^2}^2 = \lim_t \mathbb{E} M_t^2 = \lim_t \mathbb{E}\langle M\rangle_t = \mathbb{E}\langle M\rangle_\infty. \qquad (*)$$

Claim: $M\in\mathcal{M}_c^2 \Rightarrow \langle M\rangle_\infty \in L^1$

Let $T^n=\inf\{t\ge 0 : \langle M\rangle_t \ge n\}$. Then $\langle M^{T_n}\rangle_t = \langle M\rangle_{t\wedge T_n} \le n$ and since $M\in\mathcal{M}^2$ also $M^{T_n}\in\mathcal{M}^2$.

Thus by (*) applied to M^{T_n},

$$\mathbb{E} M^2_{t\wedge T_n} = \mathbb{E}\langle M\rangle_{t\wedge T_n}.$$

Take $t\to\infty$: $\mathbb{E} M^2_{t\wedge T_n} \to \mathbb{E} M^2_{T_n}$ by DCT ($\mathbb{E}\sup_t M^2_{t\wedge T_n} < \infty$)

$\mathbb{E}\langle M\rangle_{t\wedge T_n} \to \mathbb{E}\langle M\rangle_{T_n}$ by monotone convergence

Take $n\to\infty$: $\mathbb{E} M^2_{T_n} \to \mathbb{E} M^2_\infty$ by DCT ($\mathbb{E}\sup_t M^2_t < \infty$)

$\mathbb{E}\langle M\rangle_{T_n} \to \mathbb{E}\langle M\rangle_\infty$ by monotone convergence

$\Rightarrow \mathbb{E}\langle M\rangle_\infty = \mathbb{E} M^2_\infty < \infty$.

Claim: $\langle M\rangle_\infty \in L^1 \Rightarrow M \in \mathcal{M}^2$

Let $T_n = \inf\{t\geq 0 : |M_t| \geq n\}$. Again $(M^{T_n})^2 - \langle M^{T_n}\rangle$ is a a true martingale. By Fatou,

$$\mathbb{E} M^2_t \leq \liminf_{n\to\infty} \mathbb{E} M^2_{t\wedge T_n} = \liminf_{n\to\infty} \mathbb{E}\langle M\rangle_{t\wedge T_n} = \mathbb{E}\langle M\rangle_\infty.$$

Thus $M \in \mathcal{M}^2$.

2.5. Covariation

<u>Defn.</u> For M and N continuous local martingales, define the <u>covariation</u> or bracket of M and N as the process

$$\langle M, N\rangle = \frac{1}{4}\Big(\langle M+N\rangle - \langle M-N\rangle\Big).$$

Prop.

(i) $\langle M,N\rangle$ is the unique (up to indist.) finite variation process such that $MN-\langle M,N\rangle$ is a continuous local martingale.

(ii) The map $(M,N)\mapsto\langle M,N\rangle$ is bilinear and symmetric.

(iii) For every dyadic partion $(t_i^{(m)})$ with $\Delta(t^{(m)})\to 0$,

$$\langle M,N\rangle_t^{(m)}\longrightarrow\langle M,N\rangle_t \quad \text{UCP}$$

where

$$\langle M,N\rangle^{(m)}=\sum_{i=1}^{n}(M_{t_i}-M_{t_{i-1}})(N_{t_i}-N_{t_{i-1}})$$

(iv) For every stopping time T,

$$\langle M^T,N^T\rangle_t=\langle M^T,N\rangle_t=\langle M,N\rangle_{t\wedge T}.$$

(v) For $M,N\in\mathcal{M}_c^2$, $MN-\langle M,N\rangle$ is a UI martingale and

$$(M-M_0,N-N_0)_{\mathcal{M}^2}=\mathbb{E}\langle M,N\rangle_\infty.$$

Proof. Essentially the same as for $M=N$.

Example. Let B and B' be independent Brownian motions adapted with respect to the same filtration. Then BB' is a martingale, so $\langle B,B'\rangle=0$. Let $B''=\rho B+\sqrt{1-\rho^2}\,B'$ for some $\rho\in[0,1]$. Then B'' is also a BM and by bilinearity,

$$\langle B,B''\rangle_t=\rho t.$$

Prop. (Kunita–Watanabe inequality). Let M and N be continuous local martingales, and let H and K be measurable processes. Then a.s.

$$\int_0^\infty |H_s||K_s| \, |d\langle M,N\rangle_s| \leq \left(\int_0^\infty |H_s|^2 \, d\langle M\rangle_s\right)^{\frac{1}{2}} \left(\int_0^\infty |K_s|^2 \, d\langle N\rangle_s\right)^{\frac{1}{2}}. \quad \text{(KW)}$$

Proof. The idea is to approximate integrals by sums and then apply Cauchy–Schwarz. Write

$$\langle M,N\rangle_s^t = \langle M,N\rangle_t - \langle M,N\rangle_s.$$

Claim: $\forall\, 0 \leq s \leq t$, $|\langle M,N\rangle_s^t| \leq \sqrt{\langle M,M\rangle_s^t} \sqrt{\langle N,N\rangle_s^t}$ $\quad (*)$

By continuity, we can assume that s and t are dyadic rationals. In this case, in probability and thus almost surely along a subsequence,

$$\begin{aligned}
|\langle M,N\rangle_s^t| &\overset{\text{(Prop.)}}{=} \lim_{n\to\infty} \left| \sum_{i=2^n s+1}^{2^n t} (M_{2^{-n}i} - M_{2^{-n}(i-1)})(N_{2^{-n}i} - N_{2^{-n}(i-1)}) \right| \\
&\overset{\text{(CS)}}{\leq} \lim_{n\to\infty} \left| \sum_{i=2^n s+1}^{2^n t} (M_{2^{-n}i} - M_{2^{-n}(i-1)})^2 \right|^{\frac{1}{2}} \left| \sum_{i=2^n s+1}^{2^n t} (N_{2^{-n}i} - N_{2^{-n}(i-1)})^2 \right|^{\frac{1}{2}} \\
&\overset{\text{(Prop.)}}{=} |\langle M,M\rangle_s^t|^{\frac{1}{2}} \, |\langle N,N\rangle_s^t|^{\frac{1}{2}}.
\end{aligned}$$

Now fix an event on which $(*)$ holds for all $s<t$ (rational and by continuity for all irrational $s<t$ as well).

Claim: $\forall\, 0 \leq s \leq t$, $\int_s^t |d\langle M,N\rangle_u| \leq \sqrt{\langle M,M\rangle_s^t} \sqrt{\langle N,N\rangle_s^t}$

For any partition $(t_i)_{i=1}^n$ of $[s,t]$, we indeed have

$$\sum_{i=1}^{n} |\langle M,N\rangle_{t_{i-1}}^{t_i}| \overset{(*)}{\leq} \sum_{i=1}^{n} \sqrt{\langle M,M\rangle_{t_{i-1}}^{t_i}} \sqrt{\langle N,N\rangle_{t_{i-1}}^{t_i}}$$

$$\overset{(CS)}{\leq} \left(\sum_{i=1}^{n} \langle M,M\rangle_{t_{i-1}}^{t_i}\right)^{1/2} \left(\sum_{i=1}^{n} \langle N,N\rangle_{t_{i-1}}^{t_i}\right)^{1/2} = \sqrt{\langle M,M\rangle_s^t} \sqrt{\langle N,N\rangle_s^t}.$$

The claim follows by taking the sup over all partitions.

Claim: For all bounded Borel sets $B \subset [0,\infty)$,

$$\int_B |d\langle M,N\rangle_u| \leq \sqrt{\int_B d\langle M\rangle_u} \sqrt{\int_B d\langle N\rangle_u}.$$

For B a finite union of intervals, this follows from the Cauchy-Schwarz inequality as above. For general B, it then follows from a monotone class argument (exercise).

Claim: (KW) holds if $H = \sum_\ell h_\ell 1_{B_\ell}$ and $K = \sum_\ell k_\ell 1_{B_\ell}$ for disjoint bounded Borel sets B_ℓ.

$$\int |H_s K_s| \, |d\langle M,N\rangle_s| = \sum_\ell |h_\ell k_\ell| \int_{B_\ell} |d\langle M,N\rangle_s|$$

$$\leq \sum_\ell |h_\ell k_\ell| \left(\int_{B_\ell} d\langle M\rangle_s\right)^{1/2} \left(\int_{B_\ell} d\langle N\rangle_s\right)^{1/2}$$

$$\leq \left(\sum_\ell |h_\ell|^2 \int_{B_\ell} d\langle M\rangle_s\right)^{1/2} \left(\sum_\ell |k_\ell|^2 \int_{B_\ell} d\langle N\rangle_s\right)^{1/2}$$

$$= \left(\int |H_s|^2 d\langle M\rangle_s\right)^{1/2} \left(\int |K_s|^2 d\langle N\rangle_s\right)^{1/2}$$

Finally, for general H, K, approximate by H, K as above.

2.6. Semimartingales

<u>Defn.</u> A (continuous) adapted process X is a continuous semimartingale if

$$X = X_0 + M + A$$

with $X_0 \in \mathcal{F}_0$, M a (continuous) local martingale with $M_0 = 0$, and A a (continuous) finite variation process with $A_0 = 0$.

<u>Rk.</u> The decomposition is unique.

<u>Defn.</u> For $X = X_0 + M + A$ and $X' = X_0' + M' + A'$ continuous semimartingales, the quadratic variation and covariation is
$$\langle X \rangle = \langle M \rangle, \quad \langle X, X' \rangle = \langle M, M' \rangle.$$

<u>Exercise.</u> If $(t_i^{(m)})_{i=1}^{n_m}$ are nested dyadic partions of $[0,t]$,
$$\langle X, Y \rangle_t^{(m)} = \sum_{i=1}^{n_m} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}}) \longrightarrow \langle X, Y \rangle_t \quad \text{UCP}.$$

3. The Itô Integral

3.1. Simple processes

Defn. For $M \in \mathcal{M}_c^2$ and $H = \sum_{i=1}^n H_{i-1} \mathbb{1}_{(t_{i-1}, t_i]} \in \mathcal{E}$ a simple process the Itô integral is defined by

$$\int_0^t H_s \, dM_s := (H \cdot M)_t := \sum_{i=1}^n H_{i-1} (M_{t_i \wedge t} - M_{t_{i-1} \wedge t}).$$

Prop. Let $M \in \mathcal{M}_c^2$ and $H \in \mathcal{E}$. Then $H \cdot M \in \mathcal{M}_c^2$ and

$$\|H \cdot M\|_{\mathcal{M}^2}^2 = \mathbb{E}\left(\int_0^\infty H_s^2 \, d\langle M \rangle_s \right) \qquad (*) \qquad \text{(Itô isometry)}$$

Proof. Claim: $H \cdot M \in \mathcal{M}_c^2$

Let $X_t^i = H_{i-1}(M_{t_i \wedge t} - M_{t_{i-1} \wedge t})$. Since $H \cdot M = \sum_{i=1}^n X^i$, it suffices to show that $X^i \in \mathcal{M}_c^2$. Indeed, for $t \geq s$,

if $s \geq t_{i-1}$: $\mathbb{E}(X_t^i \mid \mathcal{F}_s) = H_{i-1}\big(\underbrace{\mathbb{E}(M_{t_i \wedge t} \mid \mathcal{F}_s)}_{M_s} - M_{t_{i-1}}\big) = X_s^i$

if $s < t_{i-1}$: $\mathbb{E}(X_t^i \mid \mathcal{F}_s) = \mathbb{E}\big(H_{i-1} \underbrace{\mathbb{E}(M_{t \wedge t_i} - M_{t \wedge t_{i-1}} \mid \mathcal{F}_{t_{i-1}})}_{0} \mid \mathcal{F}_s\big) = 0 = X_s^i$

so X^i is a martingale. Also, $X^i \in \mathcal{M}^2$ since

$$\|X^i\|_{\mathcal{M}^2}^2 = \sup_t \mathbb{E}(X_t^i)^2 \leq 2 \|H\|_\infty \|M\|_{\mathcal{M}^2} < \infty.$$

Claim: $\mathbb{E} X_\infty^i X_\infty^j = 0$ if $i < j$

Indeed, since $t_{j-1} \geq t_i$,

$$\mathbb{E} X^i_\infty X^j_\infty = \mathbb{E}\Big(H_{i-1} (M_{t_i} - M_{t_{i-1}}) H_{j-1} \underbrace{\mathbb{E}(M_{t_j} - M_{t_{j-1}} \mid \mathcal{F}_{t_{j-1}})}_{0} \Big) = 0$$

Claim: $\mathbb{E}(X^i_\infty)^2 = \mathbb{E}\left(\int_{t_{i-1}}^{t_i} H_s^2 \, d\langle M \rangle_s \right)$.

Indeed,

$$\mathbb{E}(X^i_\infty)^2 = \mathbb{E}\Big(H^2_{i-1} \underbrace{\mathbb{E}\big((M_{t_i} - M_{t_{i-1}})^2 \mid \mathcal{F}_{t_{i-1}}\big)}_{} \Big)$$

$$\begin{aligned} &\mathbb{E}(M^2_{t_i} - 2M_{t_{i-1}} M_{t_i} + M^2_{t_{i-1}} \mid \mathcal{F}_{t_{i-1}}) \\ &= \mathbb{E}(M^2_{t_i} - M^2_{t_{i-1}} \mid \mathcal{F}_{t_{i-1}}) \\ &= \mathbb{E}(\langle M \rangle_{t_i} - \langle M \rangle_{t_{i-1}} \mid \mathcal{F}_{t_{i-1}}) \end{aligned}$$

$$\begin{aligned} &= \mathbb{E}\big(H^2_{i-1} (\langle M \rangle_{t_i} - \langle M \rangle_{t_{i-1}}) \big) \\ &= \mathbb{E}\left(\int_{t_{i-1}}^{t_i} H_s^2 \, d\langle M \rangle_s \right) \end{aligned}$$

In summary, $\mathbb{E}(H \cdot M)^2_\infty = \sum_{i=1}^n \mathbb{E}(X^i_\infty)^2 = \mathbb{E}\left(\int_0^\infty H_s^2 \, d\langle M \rangle_s \right)$.

<u>Prop.</u> Let $M, N \in \mathcal{M}^2_c$ and $H \in \mathcal{E}$. Then

$$\langle H \bullet M, N \rangle = H \bullet \langle M, N \rangle$$

i.e.,

$$\Big\langle \underbrace{\int_0^{\cdot} H_s \, dM_s}_{\text{Itô integral}}, N \Big\rangle_t = \underbrace{\int_0^t H_s^2 \, d\langle M, N \rangle_s}_{\text{Lebesgue-Stieltjes integral}}$$

Proof. Let $H\cdot M = \sum_{i=1}^{n} X^i$ as in previous proof. Then

$$\langle X^i, N\rangle_t = H_{i-1} \langle M_{t_i\wedge\cdot} - M_{t_{i-1}\wedge\cdot}, N\rangle_t$$

$$= H_{i-1}\left(\langle M,N\rangle_{t_i\wedge t} - \langle M,N\rangle_{t_{i-1}\wedge t}\right) = \int_{t_{i-1}}^{t_i} H_s \, d\langle M,N\rangle_s$$

$$\Rightarrow \langle H\cdot M, N\rangle_t = \int_0^t H_s \, d\langle M,N\rangle_s = (H\circ\langle M,N\rangle)_t .$$

3.2. Itô isometry

Defn. For $M\in\mathcal{M}_c^2$ define $L^2(M)$ to be the space of (equivalence classes) of predictable $H:\Omega\times[0,\infty)\to\mathbb{R}$ s.t.

$$\|H\|_{L^2(M)} := \|H\|_M := \left(\mathbb{E}\left(\int_0^\infty H_s^2 \, d\langle M\rangle_s\right)\right)^{1/2} < \infty .$$

For $H,K\in L^2(M)$, set

$$(H,K)_{L^2(M)} := (H,K)_M := \mathbb{E}\left(\int_0^\infty H_s K_s \, d\langle M\rangle_s\right).$$

Fact. $L^2(M) = L^2(\Omega\times[0,\infty), \mathcal{P}, dP\,d\langle M\rangle)$ is a Hilbert space.

Prop. Let $M\in\mathcal{M}_c^2$. Then $\mathcal{E}$ is dense in $L^2(M)$.

Proof. Since $L^2(M)$ is a Hilbert space (complete!), it suffices to show that $(H,K)_M = 0$ $\forall H\in\mathcal{E}$ implies $K=0$.

So assume that $(H,K)_M = 0$ for all $H\in\mathcal{E}$ and set

$$X_t = \int_0^t K_s \, d\langle M\rangle_s .$$

X is a well-defined finite variation process since

$$\mathbb{E}\int_0^t |K_s|\, d\langle M\rangle_s \overset{(CS)}{\leq} \underbrace{\mathbb{E}\left(\int_0^t |K_s|^2\, d\langle M\rangle_s\right)^{1/2}}_{\substack{<\infty \text{ since} \\ K\in L^2(M).}} \underbrace{\left(\mathbb{E}\langle M\rangle_\infty\right)^{1/2}}_{\substack{<\infty \text{ since} \\ M\in \mathcal{M}_c^2}} < \infty$$

Claim: X is a continuous martingale.

Let $s<t$, $F\in\mathcal{F}_s$ bounded random variable, $H = F\mathbb{1}_{(s,t]}\in\mathcal{E}$.

$$\Rightarrow \quad 0 \overset{(ass)}{=} (K,H)_M = \mathbb{E}\left(F\int_s^t K_u\, d\langle M\rangle_u\right)$$

$$= \mathbb{E}\left(F(X_t - X_s)\right) = \mathbb{E}\left(F\left(\mathbb{E}(X_t\mid\mathcal{F}_s) - X_s\right)\right)$$

Since this holds for all bounded F,

$$\mathbb{E}(X_t\mid\mathcal{F}_s) = X_s \text{ a.s., so } X \text{ is a martingale.}$$

So X is a continuous local martingale and a finite variation process, so $X=0$.

$\Rightarrow$ $K_u = 0$ for $d\langle M\rangle$-a.e. u a.s.

$\Rightarrow$ $K=0$ in $L^2(M)$.

<u>Thm.</u> Let $M\in\mathcal{M}_c^2$. Then the map $H\in\mathcal{E}\mapsto H\cdot M\in\mathcal{M}_c^2$ extends uniquely to an isometry $L^2(M)\to\mathcal{M}_c^2$ (<u>the Itô isometry</u>). Moreover, $H\cdot M$ is the unique martingale in $\mathcal{M}_c^2$ s.t.

$$\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle \quad \forall N\in\mathcal{M}_c^2.$$

(Kunita-Watanabe identity)

Defn. For $M \in \mathcal{M}^2_c$ and $H \in L^2(M)$, the <u>Itô integral</u> is

$$\int_0^t H_s \, dM_s := (H \cdot M)_t .$$

Proof. For $H \in \mathcal{E}$, we have already seen that

$$\|H \cdot M\|^2_{\mathcal{M}^2} = \mathbb{E}\left(\int_0^\infty H_s^2 \, d\langle M\rangle_s\right) = \|H\|^2_{L^2(M)} .$$

Since $\mathcal{E} \subset L^2(M)$ is dense and $\mathcal{M}^2_c$ is complete, it follows that $H \mapsto H \cdot M$ extends uniquely to all of $L^2(M)$ and that the extension is an isometry.

Also, we have seen already that $\langle H \cdot M, N\rangle = H \cdot \langle M, N\rangle$ holds for $H \in \mathcal{E}$. Given $H \in L^2(M)$, choose $(H^n) \subset \mathcal{E}$ s.t. $H^n \to H$ in $L^2(M)$. Then $H^n \cdot M \longrightarrow H \cdot M$ by the first part. We will justify the following limits:

$$\begin{aligned} \langle H \cdot M, N\rangle_\infty &\overset{(\dagger)}{=} \lim_{n\to\infty} \langle H^n \cdot M, N\rangle_\infty \quad \text{in } L^1 \\ &= \lim_{n\to\infty} (H^n \cdot \langle M, N\rangle)_\infty \\ &\overset{(\dagger)}{=} (H \cdot \langle M, N\rangle)_\infty \quad \text{in } L^1 \end{aligned}$$

Indeed, $(\dagger)$ holds by the Kunita–Watanabe inequality:

$$\begin{aligned} \mathbb{E}\left|\langle H\cdot M - H^n \cdot M, N\rangle_\infty\right| &\le \left(\mathbb{E}\langle H\cdot M - H^n\cdot M\rangle_\infty\right)^{1/2} \left(\mathbb{E}\langle N\rangle_\infty\right)^{1/2} \\ &= \underbrace{\|H\cdot M - H^n \cdot M\|_{\mathcal{M}^2}} \|N\|_{\mathcal{M}^2} \\ &= \|H - H^n\|_{L^2(M)} \longrightarrow 0 . \end{aligned}$$

and
$$\mathbb{E}\left|\left((H-H^n)\cdot\langle M,N\rangle\right)_\infty\right| \le \|H-H^n\|_{L^2(M)}\cdot\|N\|_{\mathcal{M}^2}\to 0.$$

Thus $\langle H\cdot M,N\rangle_\infty = (H\cdot\langle M,N\rangle)_\infty$ and replacing N by the stopped martingale N^t gives

$$\langle H\cdot M,N\rangle_t \underset{\substack{\uparrow\\\text{property of covariation}}}{=} \langle H\cdot M,N^t\rangle_\infty \underset{\substack{\uparrow\\\text{above}}}{=} (H\cdot\langle M,N^t\rangle)_\infty \underset{\substack{\uparrow\\\text{property of covariation}}}{=} (H\cdot\langle M,N\rangle_{\cdot\wedge t})_\infty$$

$$\underset{\substack{\uparrow\\\text{property of Lebesgue-Stieltjes integral}}}{=} (H\cdot\langle M,N\rangle)_t.$$

Uniqueness: assume $X\in\mathcal{M}_c^2$ also satisfies

$$\langle X,N\rangle = H\cdot\langle M,N\rangle \qquad \forall N\in\mathcal{M}_c^2$$

$$\Rightarrow \langle X-H\cdot M,N\rangle = 0 \qquad \forall N\in\mathcal{M}_c^2$$

$$\Rightarrow \langle X-H\cdot M, X-H\cdot M\rangle = 0$$

$$\Rightarrow \|X-H\cdot M\|_{\mathcal{M}^2}=0 \Rightarrow X=H\cdot M.$$

<u>Cor.</u> If T is a stopping time, then

$$(\mathbb{1}_{[0,T]}H)\cdot M = (H\cdot M)^T = H\cdot M^T$$

<u>Proof.</u> Let $N\in\mathcal{M}_c^2$. Then

$$\langle (H\cdot M)^T, N\rangle_t = \langle H\cdot M, N\rangle_{t\wedge T} \overset{(KW)}{=} (H\cdot \langle M,N\rangle)_{t\wedge T}$$
$$= \left(H\mathbb{1}_{[0,T]}\cdot \langle M,N\rangle\right)_t$$
$$\overset{(KW)}{=} \langle H\mathbb{1}_{[0,T]}\cdot M,\ N\rangle_t$$

$\Rightarrow (H\cdot M)^T = H\mathbb{1}_{[0,T]}\cdot M$

Likewise, $\langle H\cdot M^T, N\rangle_t \overset{(KW)}{=} (H\cdot \langle M^T, N\rangle)_t = (H\cdot \langle M,N\rangle)_{t\wedge T}$, so

$$H\cdot M^T = H\mathbb{1}_{[0,T]}\cdot M.$$

<u>Cor.</u> $\langle H\cdot M, K\cdot N\rangle = (HK)\cdot \langle M,N\rangle$, i.e.,

$$\left\langle \int_0^\cdot H_s\, dM_s,\ \int_0^\cdot K_s\, dN_s\right\rangle_t = \int_0^t H_s K_s\, d\langle M,N\rangle_s.$$

<u>Proof.</u> $\langle H\cdot M, K\cdot N\rangle \overset{(KW)}{=} H\cdot \langle M, K\cdot N\rangle \overset{(KW)}{=} H\cdot (K\cdot \langle M,N\rangle)$
$$\overset{(*)}{=} (HK)\cdot \langle M,N\rangle$$

where $(*)$ is the associativity of the Lebesgue-Stieltjes integral:

$$\int_0^t h_s\, d\left(\int_0^\cdot k_u\, da_u\right)_s = \int_0^t h_s k_s\, da_s.$$

<u>Cor.</u> $\mathbb{E}\left(\int_0^t H_s\, dM_s\right) = 0$, $\mathbb{E}\left(\int_0^t H_s\, dM_s \mid \mathcal{F}_u\right) = \int_0^u H_s\, dM_s$

$$\mathbb{E}\left(\left(\int_0^t H_s\, dM_s\right)\left(\int_0^t K_s\, dN_s\right)\right) = \mathbb{E}\left(\int_0^t H_s K_s\, d\langle M,N\rangle_s\right).$$

<u>Proof.</u> $H\cdot M$ and $(H\cdot M)(N\cdot M) - \langle H\cdot K, N\cdot K\rangle$ are martingales starting at 0.

Cor. Let $H \in L^2(M)$. Then $KH \in L^2(M)$ iff $K \in L^2(H\cdot M)$ and then

$$(KH)\cdot M = K\cdot(H\cdot M).$$

Proof. Since $H^2 \cdot \langle M\rangle = \langle H\cdot M\rangle$,

$$\mathbb{E}\left(\int_0^\infty K_s^2 \underbrace{H_s^2\, d\langle M\rangle_s}_{d(H^2\cdot\langle M\rangle)}\right) = \mathbb{E}\left(\int_0^\infty K^2\, d\langle H\cdot M\rangle_s\right)$$

so $K \in L^2(H\cdot M) \Leftrightarrow KH \in L^2(M)$. Then for any $N \in \mathcal{M}_c^2$,

$$\begin{aligned}\langle (KH)\cdot M, N\rangle_t &= \left((KH)\cdot\langle M,N\rangle\right)_t \\ &= \int_0^t K_s \underbrace{H_s\, d\langle M,N\rangle_s}_{d(H\cdot\langle M,N\rangle)_s} = \left(K\cdot(H\cdot\langle M,N\rangle)\right)_t\end{aligned}$$

$$\langle K\cdot(H\cdot M), N\rangle = K\cdot\langle H\cdot M, N\rangle = K\cdot(H\cdot\langle M,N\rangle).$$

By uniqueness, the claim follows.

3.3 Itô integral for semimartingales

Defn. For M a continuous local martingale, let $L^2_{loc}(M)$ be the space of (equivalence classes of) predictable processes H s.t.

$$\int_0^t H_s^2\, d\langle M\rangle_s < \infty \quad \text{for all } t>0, \text{ a.s.}$$

A process H is locally bounded if

$$\sup_{s\le t} |H_s| < \infty \quad \text{for all } t>0, \text{ a.s.}$$

Fact. • Any continuous process is locally bounded.

• If H is locally bounded and A is a finite variation process then

$$\int_0^t |H_s|\,|dA_s| < \infty \quad \text{for all } t>0.$$

• In particular, if H is locally bounded and predictable, and M is a continuous local martingale, then $H \in L^2_{loc}(M)$.

Thm. Let M be a continuous local martingale. Then:

(i) For every $H \in L^2_{loc}(M)$ there exists a unique continuous local martingale $H\cdot M$ with $(H\cdot M)_0 = 0$ such that

$$\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle \quad \forall \text{ cont. loc. mart } N$$

(ii) If $H \in L^2_{loc}(M)$ and K is predictablen then $K \in L^2_{loc}(H\cdot M)$ if $HK \in L^2_{loc}(M)$ and then

$$H\cdot(K\cdot M) = (HK)\cdot M.$$

(iii) If T is a stopping time,

$$(1_{[0,T]}H)\cdot M = (H\cdot M)^T = H\cdot M^T.$$

Finally, if $M \in \mathcal{M}^2_c$ and $H \in L^2(M)$ then the definition of $H\cdot M$ coincides with the previous one.

Proof. (i) Assume $M_0 = 0$ and (setting $H=0$ for ω for which this fails):

$$\int_0^t H_s^2 \, d\langle M\rangle_s < \infty \quad \text{for all } t>0,\ \omega\in\Omega.$$

Set

$$T_n = \inf\left\{t\geq 0 : \int_0^t (1+H_s^2)\, d\langle M\rangle_s > n\right\}.$$

Then T_n is a stopping time, $T_n \uparrow \infty$ a.s., and

$$\langle M^{T_n}\rangle_t = \langle M\rangle_{t\wedge T_n} \leq n.$$

Thus $M^{T_n} \in \mathcal{M}_c^2$ and $\int_0^\infty H_s^2 \, d\langle M^{T_n}\rangle_s \leq n$.

$\Rightarrow H \in L^2(M^{T_n})$ and $H\cdot M^{T_n}$ is already defined,

$$H\cdot M^{T_n} = (H\cdot M^{T_m})^{T_n} \text{ for } m>n$$

Since $T_n \uparrow \infty$ a.s. there is a unique process $H\cdot M$ s.t. $(H\cdot M)^{T_n} = H\cdot M^{T_n}$ for all n. This process is adapted, continuous, and it is a local martingale since the $(H\cdot M)^{T_n}$ are martingales.

Claim: $\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle$ for all cont. loc. mart. N.

Assume $N_0=0$ and set $S_n = \inf\{t\geq 0 : |N_t|>n\}$. Then $N^{S_n}\in\mathcal{M}_c^2$ and

$$\begin{aligned}\langle H\cdot M, N\rangle^{T_n\wedge S_n} &= \langle (H\cdot M)^{T_n}, N^{S_n}\rangle = \langle H\cdot M^{T_n}, N^{S_n}\rangle \\ &= H\cdot\langle M^{T_n}, N^{S_n}\rangle = \left(H\cdot\langle M,N\rangle\right)^{T_n\wedge S_n}.\end{aligned}$$

Since $T_n \wedge S_n \uparrow \infty$ thus $\langle H\cdot M, N\rangle = H\cdot\langle M,N\rangle$. Uniqueness follows exactly as in the L^2 bounded case.

Also, (ii) & (iii) follow exactly as in the L^2 bounded case since they only use (i).

Finally, if $M \in \mathcal{M}_c^2$ and $H \in L^2(M)$ then $H\cdot M \in \mathcal{M}_c^2$ by (i) which shows $\langle H\cdot M\rangle_\infty = H^2\cdot\langle M\rangle_\infty$ and thus $\|H\cdot M\|_{\mathcal{M}^2}^2 = \mathbb{E}\langle H\cdot M\rangle_\infty < \infty$. Uniqueness in the L^2 bounded case implies that both definitions coincide.

<u>Defn.</u> Let $X = X_0 + M + A$ be a continuous semimartingale and let H be a predictable locally bounded process. The <u>Itô integral</u> $(H\cdot X)_t = \int_0^t H_s\, dX_s$ is the continuous semimartingale

$$H\cdot X = \underbrace{H\cdot M}_{\text{Itô integral}} + \underbrace{H\cdot A}_{\text{Lebesgue-Stieltjes integral}}$$

3.4. Approximation of Itô integrals

<u>Prop.</u> (stochastic DCT). Let X be a continuous semimartingale, and let H and H^n be predictable locally bounded processes, and let K be a predictable process. Let $t > 0$ and assume that a.s.

(i) $H_s^n \xrightarrow{n\to\infty} H_s$ for all $s \in [0,t]$

(ii) $|H_s^n| \leq K_s$ for all $s \in [0,t]$ and $n \in \mathbb{N}$

(iii) $\int_0^t K_s^2 \, d\langle M\rangle_s + \int_0^t |K_s| \, |dA_s| < \infty$ where $X = X_0 + A + M$

(always satisfied if K is locally bounded)

Then
$$\int_0^t H_s^n \, dX_s \to \int_0^t H_s \, dX_s.$$

<u>Proof.</u> Let $X = X_0 + A + M$. The usual DCT implies
$$\int_0^t H_s^n \, dA_s \to \int_0^t H_s \, dA_s.$$
Set $T_m = \inf\{t \geq 0 : \int_0^t K_s^2 \, d\langle M\rangle_s > m\}$. Then
$$\mathbb{E}\left(\left(\int_0^{t\wedge T_m} H_s^n \, dM_s - \int_0^{t\wedge T_m} H_s \, dM_s\right)^2\right) \underset{\substack{\uparrow \\ H \in L^2(M^{t\wedge T_m}) \Rightarrow H\cdot M^{t\wedge T_m} \in \mathcal{M}_c^2}}{=} \mathbb{E}\left(\int_0^{t\wedge T_m} (H_s^n - H_s)^2 \, d\langle M\rangle_s\right) \underset{\substack{\uparrow \\ \text{DCT}}}{\xrightarrow{n\to\infty}} 0$$

Since $T_m \wedge t = t$ eventually, a.s., the convergence holds for fixed t.

<u>Cor.</u> Let X be a continuous semimartingale, and let H be a locally bounded left-continuous process. Then for any sequence of partitions (t_i^m) of $[0,\infty)$ with $\Delta(t^m) \to 0$,
$$\lim_{m\to\infty} \sum_{i=1}^{n_m} H_{t_{i-1}^m} \left(X_{t_i^m \wedge t} - X_{t_{i-1}^m \wedge t}\right) \overset{\text{ucp}}{=} \int_0^t H_s \, dX_s.$$

<u>Proof.</u> Similar as in finite variation case using stochastic DCT.

<u>Example.</u>

$$\lim_{m\to\infty}\sum_{i=1}^{n_m} X_{t_{i-1}^m}\left(X_{t_i^m}-X_{t_{i-1}^m}\right)=\int_0^t X_s\,dX_s$$

but

$$\begin{aligned}\lim_{m\to\infty}\sum_{i=1}^{n_m} X_{t_i^m}\left(X_{t_i^m}-X_{t_{i-1}^m}\right)&=\lim_{m\to\infty}\sum_{i=1}^{n_m} X_{t_{i-1}^m}\left(X_{t_i^m}-X_{t_{i-1}^m}\right)\\&\quad+\lim_{m\to\infty}\sum_{i=1}^{n_m}\left(X_{t_i^m}-X_{t_{i-1}^m}\right)^2\\&=\int_0^t X_s\,dX_s+\langle X,X\rangle_t\end{aligned}$$

For example, if X is BM, then $\langle X,X\rangle_t=t$, so the difference is nontrivial.

<u>Rk.</u> The choice of the left endpoint gives the Itô integral. For X and Y continuous semimartingales, the <u>Stratonovich</u> integral is defined by

$$\int_0^t X_s\,\partial Y_s=\int_0^t X_s\,dY_s+\frac{1}{2}\langle X,Y\rangle_t.$$

It thus corresponds to the approximation

$$\sum_{i=1}^{n_m}\frac{1}{2}\left(X_{t_i^m}+X_{t_{i-1}^m}\right)\left(Y_{t_i^m}-Y_{t_{i-1}^m}\right)$$

Note that $\int_0^\cdot X_s\,\partial Y_s$ is generally not a local martingale.

3.5. Itô formula

<u>Lemma</u> (Integration by parts). Let X and Y be continuous semimartingales. Then a.s. for all t

$$X_tY_t - X_0Y_0 = \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + \langle X,Y\rangle_t$$

<u>Rk.</u> The term $\langle X,Y\rangle$ is called the Itô correction. It is absent if X or Y is of finite variation. Also, in terms of the Stratonovich integral,

$$X_tY_t - X_0Y_0 = \int_0^t X_s\,\partial Y_s + \int_0^t Y_s\,\partial X_s.$$

<u>Proof.</u> Clearly,

$$X_tY_t - X_sY_s = X_s(Y_t-Y_s) + Y_s(X_t-X_s) + (X_t-X_s)(Y_t-Y_s)$$

For any partition (t_i) of $[0,t]$, thus

$$\begin{aligned} X_tY_t - X_0Y_0 &= \sum_{i=1}^n \left(X_{t_i}Y_{t_i} - X_{t_{i-1}}Y_{t_{i-1}}\right) \\ &= \sum_{i=1}^n \Big(X_{t_{i-1}}(Y_{t_i}-Y_{t_{i-1}}) + Y_{t_{i-1}}(X_{t_i}-X_{t_{i-1}}) + \\ &\qquad (X_{t_i}-X_{t_{i-1}})(Y_{t_i}-Y_{t_{i-1}})\Big). \end{aligned}$$

Taking nested partitions with $\Delta(t)\to 0$, for any $t>0$, thus

$$X_tY_t - X_0Y_0 = (X\cdot Y)_t + (Y\cdot X)_t + \langle X,Y\rangle_t.$$

By continuity, this also holds for all t a.s.

Thm (Itô's formula). Let $X^1, \dots, X^p$ be continuous semimartingales, and let $f: \mathbb{R}^p \to \mathbb{R}$ be in C^2. Then a.s.

$$(*) \quad f(X_t) = f(X_0) + \sum_{i=1}^{p} \int_0^t \frac{\partial f}{\partial x^i}(X_s)\, dX^i_s + \frac{1}{2} \sum_{i,j=1}^{p} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j \rangle_s$$

where $X_t = (X^1_t, \dots, X^p_t)$.

Proof For f constant, $(*)$ is obvious.

Claim: Assume $(*)$ holds for some f. Then it also holds for g defined by $g(x) = x^k f(x)$.

x^k: kth component of x.

Indeed, apply IBP with $X = X^k$ and $Y = f(X)$:

$$g(X_t) - g(X_0) = \int_0^t X^k_s \, df(X_s) + \int_0^t f(X_s)\, dX^k_s + \langle X^k, f(X) \rangle_t .$$

By $(*)$ for f and $H \cdot (K \cdot X) = (HK) \cdot X$,

$$(X^k \cdot f(X))_t = \sum_{i=1}^{p} \int_0^t X^k_s \frac{\partial f}{\partial x^i}(X_s)\, dX^i_s + \frac{1}{2} \sum_{i,j=1}^{p} \int_0^t X^k_s \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j \rangle_s$$

By $(*)$ for f and $\langle X, H \cdot Y \rangle = H \cdot \langle X, Y \rangle$,

$$\langle X^k, f(X) \rangle_t = \sum_{i=1}^{p} \int_0^t \frac{\partial f}{\partial x^i}(X_s)\, d\langle X^k, X^i \rangle_s$$

$$\Rightarrow \quad g(X_t) = g(X_0) + \sum_{i=1}^{p} \int_0^t \frac{\partial g}{\partial x^i}(X_s)\, dX^i_s + \frac{1}{2} \sum_{i,j=1}^{p} \int_0^t \frac{\partial^2 g}{\partial x^i \partial x^j}\, d\langle X^i, X^j \rangle_s$$

By induction, $(*)$ holds for all polynomials.

Let $X^i = X^i_0 + M^i + A^i$ be the semimartingale decomposition of X.

Claim: (*) holds for all $f \in C^2$ if

$$|X_t^i(\omega)| \le n \quad \text{for all } t > 0,\ \omega \in \Omega.$$

Indeed, by the Weierstraß approximation theorem, there are polynomials p_k s.t.

$$\sup_{|x| \le n} \Big(|f(x) - p_k(x)| + |\nabla f(x) - \nabla p_k(x)| + |\nabla^2 f(x) - \nabla^2 p_k(x)| \Big) \le \frac{1}{k}.$$

Taking limits,

$$f(X_t) - f(X_0) = \lim_{k \to \infty} p_k(X_t) - p_k(X_0)$$

$$\int_0^t \frac{\partial f}{\partial x^i}(X_s)\, dX_s^i = \lim_{k \to \infty} \int_0^t \frac{\partial p_k}{\partial x^i}\, dX_s^i \quad \text{by stochastic DCT}$$

$$\int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j \rangle_s = \lim_{k \to \infty} \int_0^t \frac{\partial^2 p_k}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j \rangle_s \quad \text{by DCT.}$$

Claim: (*) holds without restriction on X.

Let $T_n = \inf\{t \ge 0 : \max_i |X_t^i| > n\}$. Then applying the above to X^{T_n},

$$f(X_{t \wedge T_n}) = f(X_0) + \sum_i \int_0^{t \wedge T_n} \frac{\partial f}{\partial x^i}(X_s)\, dX_s^i + \frac{1}{2} \sum_{i,j} \int_0^{t \wedge T_n} \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s)\, d\langle X^i, X^j \rangle_s.$$

Take $n \to \infty$.

<u>Rk.</u> In terms of the Stratonovich integral,

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x^i}(X_s) \circ dX_s^i.$$

3.6. Formal computational rules

Write $Z_t - Z_0 = \int_0^t H_s \, dX_s \iff dZ_t = H_t \, dX_t$

$Z_t - Z_0 = \langle X, Y \rangle_t = \int_0^t d\langle X, Y \rangle_s \iff dZ_t = dX_t \, dY_t$

Itô's formula:

$$df(X_t) = \sum_{i=1}^{p} \frac{\partial f}{\partial x_i} \, dX^i + \frac{1}{2} \sum_{i,j=1}^{p} \frac{\partial^2 f}{\partial x^i \partial x^j} \, dX^i \, dX^j$$

For Brownian motion, $(dX)^2 = d\langle B \rangle = dt$. Thus

$$df(B_t) = f'(X) \, dX + \frac{1}{2} f''(X) \, dt.$$

Associativity:

$$H_t(K_t \, dX_t) = (H_t K_t) \, dX_t \iff H \bullet (K \bullet X) = (HK) \bullet X$$

Kunita–Watanabe identity:

$$H_t \, dX_t \, dY_t = (H_t \, dX_t) \, dY_t \iff H \bullet \langle X, Y \rangle = \langle H \bullet X, Y \rangle$$

Integration by parts:

$$d(X_t Y_t) = X_t \, dY_t + Y_t \, dX_t + dX_t \, dY_t$$

4. Applications to Brownian motion and martingales

4.1. Levy's characterisation of Brownian motion

Thm. Let $X=(X^1,\dots,X^d)$ be continuous local martingales, satisfying $X_0=0$ and $\langle X^i,X^j\rangle = \delta_{ij}t$ for all $t\geq 0$. Then X is a standard d-dimensional Brownian motion.

Exercise. It suffices to show that

$$\mathbb{E}\left(e^{i\theta\cdot(X_t-X_s)}\,\middle|\,\mathcal{F}_s\right)=e^{-\frac{1}{2}|\theta|^2(t-s)} \quad \text{for all } s<t,\ \theta\in\mathbb{R}^d.$$

Proof of thm. Fix $\theta\in\mathbb{R}^d$ and set $Y_t=\theta\cdot X_t=\sum_{i=1}^d \theta^i X^i_t$. Then

$$\langle Y\rangle_t=\langle Y,Y\rangle_t=\sum_{i,j}\theta^i\theta^j\langle X^i,X^j\rangle_t=|\theta|^2 t \quad \text{by assumption.}$$

Let $Z_t=e^{iY_t+\frac{1}{2}\langle Y\rangle_t}=e^{iY_t+\frac{1}{2}|\theta|^2 t}$. By Itô's formula applied to $X=iY+\frac{1}{2}\langle Y\rangle_t$ and $f(x)=e^x$,

$$dZ_t=df(X_t)=Z_t\left(i\,dY_t+\tfrac{1}{2}d\langle Y\rangle_t-\tfrac{1}{2}d\langle Y\rangle_t\right)=iZ_t\,dY_t$$

i.e.,

$$Z_t-Z_0=i\int_0^t Z_t\,dY_t=i(Z\cdot Y)_t.$$

Thus Z is a local martingale. Since Z is also bounded, it is in fact a (true) martingale. Hence

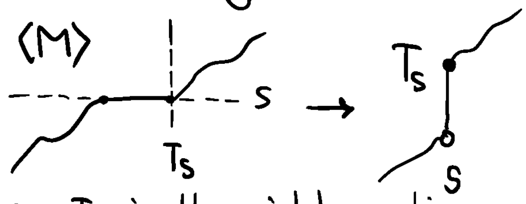
$$\mathbb{E}(Z_t|\mathcal{F}_s)=Z_s \;\Rightarrow\; \mathbb{E}\left(e^{i(Y_t-Y_s)}\,\middle|\,\mathcal{F}_s\right)=e^{-\frac{1}{2}|\theta|^2(t-s)}$$

which is the claim.

4.2. Dubins–Schwarz Theorem

Thm. Let M be a continuous local martingale with $M_0 = 0$ and $\langle M\rangle_\infty = \infty$. Let

$$T_s = \inf\{t \geq 0 : \langle M\rangle_t > s\},$$
$$B_s = M_{T_s}, \quad \mathcal{G}_s = \mathcal{F}_{T_s}.$$



$s \mapsto T_s$ is the right-continuous inverse of $t \mapsto \langle M\rangle_t$.

Then

$(\mathcal{G}_s)$ is a filtration satisfying the usual conditions,
(B_s) is a standard Brownian motion adapted to $(\mathcal{G}_s)$,
M is a (random) time change of B: $M_t = B_{\langle M\rangle_t}$.

Lemma. Almost surely, for all $u < v$,

$$M \text{ is constant on } [u,v] \Leftrightarrow \langle M\rangle \text{ is constant on } [u,v].$$

Proof. By continuity and since $\langle M\rangle$ is increasing, it suffices to show that for all fixed $u < v$,

$$\{M_t = M_u \ \forall t \in [u,v]\} = \{\langle M\rangle_u = \langle M\rangle_v\} \quad \text{a.s.}$$

Let $N_t = M_t - M_{t\wedge u} = \int_{t\wedge u}^{t} dM_s$. Then $\langle N\rangle_t = \int_{t\wedge u}^{t} d\langle M\rangle_s = \langle M\rangle_t - \langle M\rangle_{t\wedge u}$.

Let $T_\varepsilon = \inf\{t \geq 0 : \langle N\rangle_t > \varepsilon\}$. Then $N^{T_\varepsilon} \in \mathcal{M}_c^2$ since $\langle N^{T_\varepsilon}\rangle_\infty \leq \varepsilon$ and $\mathbb{E}(N_t^{T_\varepsilon})^2 \leq \mathbb{E}\langle N^{T_\varepsilon}\rangle_\infty \leq \varepsilon$.

$$\Rightarrow \mathbb{E}\left(1_{\{\langle M\rangle_v = \langle M\rangle_u\}} N_t^2\right) = \mathbb{E}\left(1_{\langle N\rangle_v = 0} N_t^2\right)$$
$$= \mathbb{E}\left(1_{\langle N\rangle_v = 0} N_{t\wedge T_\varepsilon}^2\right) \leq \varepsilon \quad \text{for } t \in [u,v]$$

$\Rightarrow N_t = 0$ a.s. on $\{\langle M\rangle_v = \langle M\rangle_u\}$ for $t \in [u,v]$.

Thus $\langle M\rangle_v = \langle M\rangle_u$ implies M is constant on $[u,v]$ a.s.

Other direction: exercise (for example by approximation).

<u>Lemma.</u> B is continuous.

<u>Proof.</u> Since T_s is càdlàg and M is continuous, $B_s = M_{T_s}$ is right-continuous. Left-continuity is equivalent to

$$B_s = B_{s-} \Leftrightarrow M_{T_s} = M_{T_{s^-}} \text{ where } T_{s-} = \inf\{t \geq 0 : \langle M\rangle_t \geq s\}.$$

Thus if $T_s = T_{s-}$ then B is left-continuous at s.
On the other hand, if $T_s > T_{s-}$ then $\langle M\rangle$ is constant on $[T_{s-}, T_s]$, a.s., and by the lemma M is constant also, so again $M_{T_s} = M_{T_{s-}}$.

<u>Lemma.</u> $(\mathcal{G}_s)$ is a filtration obeying the usual conditions, $\mathcal{G}_\infty = \mathcal{F}_\infty$, and (B_s) is adapted to it.

<u>Proof.</u>
$$\begin{aligned} A \in \mathcal{G}_s &\Leftrightarrow A \cap \{T_s \leq u\} \in \mathcal{F}_u \ \forall u \\ &\Rightarrow A \cap \{T_t \leq u\} = A \cap \{T_s \leq u\} \cap \{T_t \leq u\} \in \mathcal{F}_u \ \forall u \\ &\Leftrightarrow A \in \mathcal{G}_t \quad \text{if } t \geq s \end{aligned}$$

So $(\mathcal{G}_s)$ is a filtration. Right-continuity follows from that of $(\mathcal{F}_t)$ and $s \mapsto T_s$, and completeness from that of $(\mathcal{F}_t)$.

To see that $B_s \in \mathcal{G}_s = \mathcal{F}_{T_s}$ we apply the following fact from AP: if X is càdlàg and T a stopping time then $X 1_{T<\infty} \in \mathcal{F}_T$.

Lemma. B is a martingale with respect to $(\mathcal{G}_s)$ and $\langle B\rangle_s = s$.

Proof. Since $\langle M^{T_s}\rangle_\infty = \langle M\rangle_{T_s} = s < \infty$, $M^{T_s} \in \mathcal{M}_c^2$. Thus $(M^2 - \langle M\rangle)^{T_s}$ is a UI martingale. Hence, by OST, for $r<s$,

$$\mathbb{E}(B_s \mid \mathcal{G}_r) = \mathbb{E}(M_\infty^{T_s} \mid \mathcal{F}_{T_r}) = M_{T_r} = B_r.$$

$$\mathbb{E}(B_s^2 - s \mid \mathcal{G}_r) = \mathbb{E}((M^2 - \langle M\rangle)^{T_s} \mid \mathcal{F}_{T_r}) = M_{T_r}^2 - \langle M\rangle_{T_r} = B_r^2 - r.$$

$\Rightarrow$ B is a continuous martingale with $\langle B\rangle_s = s$.

Proof of thm. By Levy's characterisation, the lemmas imply that B is a standard Brownian motion. They also imply the other assertions.

4.3 Girsanov Theorem

Example. Let X be an n-dimensional centred Gaussian vector with covariance matrix $C = (C_{ij})$:

$$\mathbb{E}f(X) = \left(\det \tfrac{M}{2\pi}\right)^{1/2} \int_{\mathbb{R}^n} f(x)\, e^{-\frac{1}{2}(x, Mx)}\, dx, \quad M = C^{-1}.$$

$$\Rightarrow \quad \mathbb{E}f(X+a) = \left(\det \tfrac{M}{2\pi}\right)^{1/2} \int_{\mathbb{R}^n} f(x)\, \underbrace{e^{-\frac{1}{2}(x-a, M(x-a))}}_{e^{-\frac{1}{2}(x,Mx)}\, \underbrace{e^{-\frac{1}{2}(a,Ma)+(x,Ma)}}_{Z(x)}}\, dx$$

$$= \mathbb{E}(Z f(X))$$

Thus if $X \sim \mathcal{N}(0,C)$ under a measure $\mathbb{P}$ then $X \sim \mathcal{N}(a,C)$ under the measure $\mathbb{Q}$ where $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z$.

__Example.__ Let B be a standard Brownian motion and fix $0=t_0<\cdots<t_n$. Then $(B_{t_i})_{i=1}^n$ is a centred Gaussian vector with

$$\mathbb{E}f((B_{t_i})) = \text{const.} \int_{\mathbb{R}^n} f(x)\, e^{-\frac{1}{2}\sum_{i=1}^n \frac{(x_i-x_{i-1})^2}{t_i-t_{i-1}}}\, dx_1 \cdots dx_n$$

For any deterministic function $h: \mathbb{R}_+ \to \mathbb{R}$,

$$\mathbb{E} f((B+h)_{t_i}) = \mathbb{E}(Z\, f(B)),$$

$$Z = \exp\left(-\frac{1}{2}\sum_{i=1}^n \frac{(h_{t_i}-h_{t_{i-1}})^2}{t_i-t_{i-1}} + \sum_{i=1}^n \frac{(h_{t_i}-h_{t_{i-1}})(B_{t_i}-B_{t_{i-1}})}{t_i-t_{i-1}}\right)$$

__Defn.__ Let M be a continuous local martingale. The __stochastic exponential__ of M is

$$\mathcal{E}(M) = e^{M_t - \frac{1}{2}\langle M\rangle_t}.$$

__Fact.__ If $M_0=0$, $Z=\mathcal{E}(M)$ is a continuous local martingale and

$$dZ_t = Z_t\, dM_t, \quad \text{i.e.,} \quad Z_t = 1+\int_0^t Z_s\, dM_s.$$

__Proof.__ Apply Itô's formula.

__Thm__ (Girsanov). Let L be a continuous local martingale with $L_0=0$. Suppose that $\mathcal{E}(L)$ is a UI martingale. Suppose

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(L)_\infty.$$

Then if M is a continuous local martingale w.r.t. $\mathbb{P}$, then $\tilde{M} = M - \langle M,L\rangle$ is one w.r.t. $\mathbb{Q}$.

<u>Rk.</u> $\langle \tilde{M} \rangle = \langle M \rangle$.

<u>Proof.</u> Let $T_n = \inf\{t \geq 0 : |\tilde{M}_t| \geq n\}$. Then T_n is a stopping time and $\mathbb{P}(T_n \uparrow \infty) = 1$ by continuity of $\tilde{M}$. Since $\mathbb{Q} \ll \mathbb{P}$, also $\mathbb{Q}(T_n \uparrow \infty) = 1$. Thus it suffices to show that $\tilde{M}^{T_n}$ is a continuous local martingale w.r.t. $\mathbb{Q}$.

Claim: if $Y = M^{T_n} - \langle M^{T_n}, L \rangle$ and $Z = \mathcal{E}(L)$ then YZ is a continuous local martingale w.r.t. $\mathbb{P}$.

$$d(ZY) \overset{(\text{Itô})}{=} Y_t \, dZ_t + Z_t \, dY_t + d\langle Z, Y \rangle_t$$

$$= Y_t Z_t \, dL_t + Z_t (dM_t^{T_n} - d\langle M^{T_n}, L \rangle_t) + Z_t \, d\langle M^{T_n}, L \rangle_t$$

| since $d\langle Z, Y \rangle = d\langle Z, M^{T_n} \rangle \underset{\uparrow}{=} Z d\langle L, M^{T_n} \rangle$

$dZ = Z dL$ and $\langle Z \cdot L, M^{T_n} \rangle = Z \cdot \langle L, M^{T_n} \rangle$

$$= Y_t Z_t \, dL_t + Z_t \, dM_t^{T_n}$$

Since L and M^{T_n} are local martingales, so is ZY.

Claim: ZY is a UI martingale.

This follows from the fact that $Z = \mathcal{E}(L)$ is a UI martingale by assumption and $|Y| \leq n$ is bounded. Indeed, UI is stable under multiplication by bounded random variables, so ZY is UI. To see it is a martingale, recall this is equivalent to

$\forall t : \mathcal{X}_t = \{M_T : T \text{ is a stopping time}, T \leq t\}$ is UI.

Claim: Y is a martingale w.r.t. $\mathbb{Q}$

Indeed, since Z and ZY are UI martingales,

$$\mathbb{E}^{\mathbb{Q}}(Y_t - Y_s \mid \mathcal{F}_s) = \mathbb{E}^{\mathbb{P}}(Z_\infty Y_t - Z_\infty Y_s \mid \mathcal{F}_s)$$
$$= \mathbb{E}^{\mathbb{P}}(Z_t Y_t - Z_s Y_s \mid \mathcal{F}_s) = 0.$$

Thus $Y = (M - \langle M, L\rangle)^{T_n}$ is martingale and $T_n \uparrow \infty$ a.s. Hence $M - \langle M, L\rangle$ is a local martingale.

<u>Cor</u> Let B be a standard Brownian motion under $\mathbb{P}$, and let L be a continuous local martingale with $L_0 = 0$ such that $\mathcal{E}(L)$ is UI. Then $\tilde{B} = B - \langle B, L\rangle$ is a standard Brownian motion under $\mathbb{Q}$.

<u>Proof</u> By Girsanov's Theorem, $\tilde{B}$ is a continuous local martingale. Since

$$\langle \tilde{B}\rangle_t = \langle B\rangle_t = t,$$

by Levy's characterisation, $\tilde{B}$ is a standard Brownian motion.

<u>Prop.</u> Suppose $\langle L\rangle$ is bounded, i.e., $\langle L\rangle_\infty \le C$. Then

$$\mathbb{P}\left(\sup_{t \ge 0} L_t \ge a\right) \le e^{-a^2/2C}. \qquad (*)$$

and in particular $\mathcal{E}(L)$ is a UI martingale.

<u>Proof</u> Let $T = \inf\{t \geq 0 : L_t \geq a\}$ and $\theta \in \mathbb{R}$. Then

$$Z_t = e^{\theta L^T_t - \frac{1}{2}\theta^2 \langle L^T \rangle_t}$$

is a bounded martingale. Hence $\mathbb{E} Z_\infty = \mathbb{E} Z_0 = 1$. Thus

$$\mathbb{P}(\sup_t L_t \geq a) \leq \mathbb{P}(L^T_\infty \geq a) = \mathbb{P}(Z_\infty \geq e^{\theta a - \frac{1}{2}\theta^2 C})$$
$$\leq e^{-\theta a + \frac{1}{2}\theta^2 C}$$

Taking the inf over θ gives $(*)$.

Consequently, $\mathbb{E}(\sup_t \mathcal{E}(L)_t) \leq \mathbb{E}(\exp(\sup_t L_t))$

$$= \int_0^\infty \mathbb{P}(\exp(\sup_t L_t) \geq \lambda)\, d\lambda$$
$$= 1 + \int_1^\infty e^{-(\log \lambda)^2 / 2C}\, d\lambda < \infty.$$

Thus $\mathcal{E}(L)$ is bounded by $\sup_t \mathcal{E}(L)_t \in L^1$. This implies $\mathcal{E}(L)$ is a UI martingale.

A more general criterion is Novikov's condition.

<u>Thm</u> (Novikov). Let M be a continuous local martingale with $M_0 = 0$. Then

$$\mathbb{E}(e^{\frac{1}{2}\langle M \rangle_\infty}) < \infty$$

implies that $\mathcal{E}(M)$ is a UI martingale.

Example. Consider the SDE

$$dX_t = b(X_t)\,dt + dB_t\,, \qquad t \le T$$

One can construct a weak solution (see later) as follows. Let X be a standard Brownian motion under $\mathbb{P}$. Set

$$L_t = \int_0^{t\wedge T} b(X_s)\,dX_s.$$

Assume that $\mathcal{E}(L)$ is a UI martingale. This is for example true if b is bounded. Then

$$X_t - \langle X, L\rangle_t = X_t - \int_0^{t\wedge T} b(X_s)\,d\langle X\rangle_s = X_t - \int_0^{t\wedge T} b(X_s)\,ds$$

is a standard Brownian motion under $\mathbb{Q}$ where $d\mathbb{Q}/d\mathbb{P} = \mathcal{E}(L)_\infty$. Denote $B = X - \langle X, L\rangle$. Then

$$X_t = \int_0^{t\wedge T} b(X_s)\,ds + B_t.$$

4.4. The Cameron-Martin formula

Defn. The Wiener space $(W, \mathcal{W}, P)$ is given by $W = C(\mathbb{R}_+, \mathbb{R})$, $\mathcal{W} = \sigma(X_t : t \ge 0)$ where $X_t : W \to \mathbb{R}$ is given by $X_t(\omega) = \omega(t)$, and P is the unique probability measure on $(W, \mathcal{W})$ that makes X a standard Brownian motion. This setup is also called the canonical version of Brownian motion.

Defn. The Cameron–Martin space is

$$\mathcal{H} = \{h \in W : h(t) = \int_0^t g(s)\, ds \text{ for some } g \in L^2(\mathbb{R}_+)\}.$$

For $h \in \mathcal{H}$ the function $g = \dot{h}$ is the weak derivative of h.

Rk. $\mathcal{H}$ defines a Hilbert space with inner product

$$(h,f)_{\mathcal{H}} = \int_0^\infty \dot{h}(s)\dot{f}(s)\, ds.$$

Its dual space can be identified with

$$\mathcal{H}^* = \{\mu \in M(\mathbb{R}_+) : \int_0^\infty (s \wedge t)\mu(ds)\mu(dt) = (\mu,\mu)_{\mathcal{H}^*} < \infty,\ \mu(\{0\}) = 0\},$$

i.e. for $\ell : \mathcal{H} \to \mathbb{R}$ bounded linear, $\ell(h) = \int_0^\infty h(t)\mu(dt)$ for some μ.

Thm (Cameron–Martin). Let $h \in \mathcal{H}$ and define P^h by $P^h(A) = P(\{w \in W : w + h \in A\})$ for $A \in \mathcal{W}$. Then

$$\frac{dP^h}{dP} = \exp\left(\int_0^\infty \dot{h}(s)\, dX_s - \frac{1}{2}\int_0^\infty \dot{h}(s)^2\, ds\right).$$

Proof. Apply Girsanov with $L_t = \int_0^t \dot{h}(s)\, dX_s$. Since

$$\langle L \rangle_\infty = \int_0^\infty \dot{h}(s)^2\, ds = \|h\|_{\mathcal{H}}^2 < \infty,$$

$\mathcal{E}(L)$ is a UI martingale.

Rk. Intuitively, Brownian motion should be the standard Gaussian measure on $\mathcal{H}$. This does not exist, but the Cameron–Martin formula gives a way to interpret this.

4.5. Burkholder–Davis–Gundy inequalities

Let $M_t^* = \sup_{s \le t} |M_s|$.

<u>Thm.</u> For $p > 0$, there are $c_p, C_p > 0$ such that for every continuous local martingale M with $M_0 = 0$, and every stopping time T,

$$c_p \mathbb{E} \langle M \rangle_T^{p/2} \le \mathbb{E} |M_T^*|^p \le C_p \mathbb{E} \langle M \rangle_T^{p/2}.$$

<u>Proof.</u> Example sheet.

5. Stochastic Differential Equations

5.1. Notions of solutions

Defn. Let $d, m \in \mathbb{N}$, $b: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, $\sigma: \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be locally bounded. A weak solution to the SDE

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dB_t \qquad (E(\sigma, b))$$

consists of

- a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ obeying the usual conditions;
- an m-dimensional $(\mathcal{F}_t)$-Brownian motion;
- an $(\mathcal{F}_t)$-adapted continuous $\mathbb{R}^d$-valued process such that

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\,dB_s + \int_0^t b(s, X_s)\,ds. \qquad (*)$$

Defn. For a strong solution to $E(\sigma, b)$, we specify the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Brownian motion B, and choose $(\mathcal{F}_t)$ to be the completed filtration generated by B. Then a strong solution is an $(\mathcal{F}_t)$-adapted process X such that $(*)$ holds.

Rk. The completed (also called augmented) filtration is s.t. $\mathcal{F}_0$ contains all $(\mathcal{F}_\infty, \mathbb{P})$-nullsets. It can be shown that it is in fact right-continuous (→ Karatzas-Shreve, Prop. 7.7.)

<u>Defn.</u> For the SDE, we say that there is

- <u>uniqueness in law</u> (or weak uniqueness) if all weak solutions to $E(\sigma,b)$ with $X_0 = x \in \mathbb{R}^d$ have the same law.
- <u>pathwise uniqueness</u> (or strong uniqueness) if, for fixed $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and B, all solutions to $E(\sigma, b)$ are indistinguishable.

<u>Example.</u> (Tanaka) The SDE

$$(*) \quad dX_t = \text{sign}(X_t)\, dB_t, \quad X_0 = x, \quad \text{where } \text{sign}(x) = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

has a weak solution and weak uniqueness holds, but pathwise uniqueness fails.

<u>Proof.</u> Let X be a standard BM with $X_0 = x$. Set

$$B_t = \int_0^t \text{sign}(X_s)\, dX_s.$$

$$\Rightarrow \quad x + \int_0^t \text{sign}(X_s)\, dB_s = x + \int_0^t \underbrace{\text{sign}(X_s)^2}_{1}\, dX_s = X_t$$

Thus $dX_t = \text{sign}(X_t)\, dB_t$, $X_0 = x$. Moreover, B is a standard Brownian motion since it is a local martingale and $\langle B \rangle_t = \int_0^t d\langle X_s \rangle = t$.

In fact, by the same argument, any solution is a standard BM. So uniqueness in law holds.

Claim: if X is a solution with $X_0=0$, then $-X$ is also one. In particular, pathwise uniqueness fails.

Indeed, $-X_t = -\int_0^t \operatorname{sign}(X_s)\,dB_s$

$$= \int_0^t \operatorname{sign}(-X_s)\,dB_s + 2\underbrace{\int_0^t 1_{X_s=0}\,dB_s}_{N_t}$$

Claim: $N=0$

Indeed, N is a continuous local martingale and

$$\langle N\rangle_t = \int_0^t 1_{X_s=0}\,ds = 0$$

since the zero set of Brownian motion has Lebesgue measure 0.

Since $N=0$, both X and $-X$ solve $(*)$ and hence pathwise uniqueness fails.

<u>Rk.</u> X is not a strong solution.

<u>Thm.</u> (Yamada-Watanable) Assume $E(\sigma,b)$ has a weak solution with $X_0=x$ and that pathwise uniqueness holds. Then uniqueness in law holds and for any $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$ and B there is a unique strong solution to $E(\sigma,b)$.

Proof omitted (result will not be used).

Thm. Suppose that b and σ are locally Lipschitz, in the sense that there are $K_n > 0$ such that for all $t \geq 0$ and $|x|, |y| \leq n$,

$$|b(t,x) - b(t,y)| \leq K_n |x-y|$$
$$|\sigma(t,x) - \sigma(t,y)| \leq K_n |x-y|.$$

Then pathwise uniqueness holds for $E(\sigma,b)$.

Proof. Let X and $\hat{X}$ be two solutions to $E(\sigma,b)$ defined on the same probability space such that $X_0 = \tilde{X}_0$ a.s.

Let
$$T_n = \inf\{t \geq 0 : |X_t| \geq n \text{ or } |\tilde{X}_t| \geq n\},$$
$$f_n(t) = \mathbb{E}\left(|X_{t\wedge T_n} - \tilde{X}_{t\wedge T_n}|^2\right).$$

By continuity of X and $\tilde{X}$ it suffices to show that for all n and t, one has $f_n(t) = 0$.

By Itô's formula,

$$|X_{t\wedge T_n} - \tilde{X}_{t\wedge T_n}|^2 = \int_0^{t\wedge T_n} \underbrace{2(X_s - \tilde{X}_s)\cdot(b(s,X_s) - b(s,\tilde{X}_s))}_{\leq K_n |X_s - \tilde{X}_s|^2} ds$$
$$+ \int_0^{t\wedge T_n} 2(X_s - \tilde{X}_s)\cdot(\sigma(s,X_s) - \sigma(s,\tilde{X}_s))\, dB_s$$
$$+ \int_0^{t\wedge T_n} 2\underbrace{|\sigma(s,X_s) - \sigma(s,\tilde{X}_s)|^2}_{\leq K_n^2 |X_s - \tilde{X}_s|^2} ds$$

Since $(X_s - \tilde{X}_s)\cdot(\sigma(s,X_s) - \sigma(s,\tilde{X}_s))$ is bounded on $s \le T_n$, the middle term is a mean 0 martingale. Thus

$$f_n(t) \le 2(K_n + K_n^2)\int_0^t f_n(s)\,ds$$

Since $f_n(0)=0$ and f_n is bounded, Gronwall's inequality implies $f_n(t)=0$ for all t and n.

Gronwall's inequality. (→ Example sheet). Let $T>0$ and let $f:[0,T]\to[0,\infty)$ be a bounded Borel function. Then

$$f(t) \le a + b\int_0^t f(s)\,ds \qquad \text{for all } t\le T$$

implies

$$f(t) \le a e^{bt} \quad \text{for all } t \le T.$$

Rk. The proof of the theorem also shows that solutions defined up to time T must agree up to time T.

5.2. Strong existence for Lipschitz coefficients

Thm. Assume b and σ are globally Lipschitz, i.e., there is $K>0$ such that for all $x,y\in\mathbb{R}^d$ and $t\ge 0$,

$$|b(t,x)-b(t,y)| \le K\,|x-y|$$
$$|\sigma(t,x)-\sigma(t,y)| \le K\,|x-y|.$$

For any $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$, any $(\mathcal{F}_t)$-Brownian motion B, any $x\in\mathbb{R}^d$, there is a strong solution to $E(\sigma,b)$ with $X_0=x$.

<u>Proof.</u> To simplify notation, assume $d=m=1$. Define

$$F(X)_t = x + \int_0^t \sigma(s,X_s)\, dB_s + \int_0^t b(s,X_s)\, ds.$$

Given any $(\mathcal{F}_t)$ we will find an $(\mathcal{F}_t)$-adapted X such that $F(X)=X$. Such a fixed point is a strong solution since we can take $(\mathcal{F}_t)$ to be the filtration induced by B.

The proof is by Picard iteration. Let $T>0$. For X continuous, adapted, set

$$|||X|||_T = \mathbb{E}\Big(\sup_{t\le T} |X_t|^2\Big)^{1/2}$$

Then $\mathcal{B} = \{ X : \Omega\times[0,T]\to\mathbb{R} \text{ continuous adapted} : |||X|||_T < \infty \}$ is a Banach space.

Claim: $|||F(X)-F(Y)|||_T^2 \le (2T+8)K^2 \int_0^T |||X-Y|||_t^2\, dt.$

Indeed,
$$|||F(X)-F(Y)|||_T^2 \overset{(a+b)^2\le 2a^2+2b^2}{\le} 2\, \mathbb{E}\Big(\sup_{t\le T}\Big|\int_0^t (b(s,X_s)-b(s,Y_s))\, ds\Big|^2\Big) + 2\, \mathbb{E}\Big(\sup_{t\le T}\Big|\int_0^t (\sigma(s,X_s)-\sigma(s,Y_s))\, dB_s\Big|^2\Big)$$
$$= (\mathrm{I}) + (\mathrm{II})$$

and
$$(\mathrm{I}) \overset{(CS)}{\le} 2T\, \mathbb{E}\Big(\int_0^T |b(s,X_s)-b(s,Y_s)|^2\, ds\Big) \le 2TK^2 \int_0^T |||X-Y|||_t^2\, dt$$

$$(\mathrm{II}) \overset{(\text{Doob}+\text{Itô})}{\le} 8\, \mathbb{E}\Big(\int_0^T |\sigma(s,X_s)-\sigma(s,Y_s)|^2\, ds\Big) \le 8K^2 \int_0^T |||X-Y|||_t^2\, dt$$

$$\mathbb{E}\sup_{t\le T}|M_t|^2 \le 4\,\mathbb{E} M_T^2 = 4\,\mathbb{E}\langle M\rangle_T.$$

Claim: $|||F(0)|||_T < \infty$

$$F(0)_t = x + \int_0^t b(s,0)\,ds + \int_0^t \sigma(s,0)\,dB_s$$

$$\Rightarrow |||F|||_T^2 \leq 3\Big(|x|^2 + \underbrace{|||\int_0^t b(s,0)\,ds\,|||_T^2}_{\leq T\int_0^T |b(s,0)|^2 ds} + \underbrace{|||\int_0^t \sigma(s,0)\,dB_s\,|||_T^2}_{\leq 4\,\mathbb{E}\left(\int_0^T |\sigma(s,0)|^2 ds\right)}\Big) < \infty$$

Let $X_t^0 = 0$ and $X^{i+1} = F(X^i)$. Then

$$|||X^{i+1} - X^i|||_T^2 \leq C\int_0^T |||X^i - X^{i-1}|||_t^2\,dt$$

$$\leq C^2 \int_0^T\int_0^t |||X^{i-1} - X^{i-2}|||_s^2\,ds$$

$$\leq C^n \underbrace{\int_0^T\int_0^{t_{n-1}} \cdots \left(\int_0^{t_1} dt_0\right) dt_1 \cdots dt_{n-1}}_{\frac{T^n}{n!}} \underbrace{|||X^1 - X^0|||_T^2}_{\|F(0)\|_T}$$

$\Rightarrow \sum_{i=1}^{\infty} |||X^{i+1} - X^i|||_T < \infty$

$\Rightarrow X^i$ converges uniformly on $[0,T]$, a.s.

$\Rightarrow F(X) = X$ on $[0,T]$

By uniqueness, solutions defined using different T must agree when both are defined. Hence we can find a solution defined on all of $[0,\infty)$.

<u>Rk.</u> When $\sigma = 1$, the integral equation $X_t = X_0 + \int_0^t b(s,X_s)\,ds + B_t$ can be solved for any continuous function by the same argument replacing $|||\cdot|||_T$ by $\|\cdot\|_T = \sup_{t\leq T} |X_t|$.

5.3. The solution map

The following proposition provides a rough estimate on the dependence of the solution on the initial condition.

Prop. Under the same assumptions as in the theorem, let X^x be the solution with initial condition $X_0^x = x$. For $p \geq 2$,

$$\mathbb{E}\left(\sup_{s\leq t} |X_s^x - X_s^y|^p\right) \leq C_{p,t}\, |x-y|^p$$

Proof. Fix $x,y \in \mathbb{R}^d$ and let $T = \inf\{t \geq 0 : |X_t^x| \geq n \text{ or } |X_t^y| \geq n\}$. Since $|a+b+c|^p \leq 3^{p-1}(|a|^p + |b|^p + |c|^p)$,

$$\mathbb{E}\left(\sup_{s\leq t} |X_{s\wedge T}^x - X_{s\wedge T}^y|^p\right)$$

$$\leq 3^{p-1}\Bigg[|x-y|^p + \underbrace{\mathbb{E}\left(\sup_{s\leq t}\left|\int_0^{s\wedge T}(\sigma(r,X_r^x) - \sigma(r,X_r^y))\, dB_r\right|^p\right)}_{(A)}$$

$$+ \underbrace{\mathbb{E}\left(\sup_{s\leq t}\left|\int_0^{s\wedge T}(b(r,X_r^x) - b(r,X_r^y))\, dr\right|^p\right)}_{(B)}\Bigg]$$

where $(A) \overset{(BDG)}{\leq} C_p \mathbb{E}\left(\left(\int_0^{t\wedge T} |\sigma(r,X_r^x) - \sigma(r,X_r^y)|^2\, dr\right)^{p/2}\right)$

$$\overset{(\text{Höder})}{\leq} C_p t^{\frac{p}{2}-1}\, \mathbb{E}\left(\int_0^{t\wedge T} |\sigma(r,X_r^x) - \sigma(r,X_r^y)|^p\, dr\right)$$

$$(B) \overset{(\text{Höder})}{\leq} C_p t^{p-1}\, \mathbb{E}\left(\int_0^{t\wedge T} |b(r,X_r^x) - b(r,X_r^y)|^p\, dr\right)$$

Using the Lipschitz condition for σ and b, therefore there are $C_{p,t}$ such that

$$\underbrace{\mathbb{E}\left(\sup_{s\le t}|X^x_{s\wedge T}-X^y_{s\wedge T}|^p\right)}_{f(t)} \le 3^{p-1}|x-y|^p + C_{p,t}\int_0^t \underbrace{\mathbb{E}|X^x_{s\wedge T}-X^y_{s\wedge T}|^p}_{\le f(s)}\,ds .$$

Note that f is bounded because of the stopping time T. By Gronwall, thus

$$f(t)\le 3^{p-1}|x-y|^p\exp(C_{p,t}t).$$

By Fatou, we can take $n\to\infty$ and get the claim.

Strong solutions are functions of Brownian motion in the following sense. Recall the d-dimensional Wiener space $(W^d,\mathcal{W}^d,P)$ where $W^d=C(\mathbb{R}_+,\mathbb{R}^d)$. The space W^d can be given the topology of uniform convergence on compact intervals, which is induced by the metric

$$d(w,\tilde{w})=\sum_{k=1}^{\infty}\alpha_k\left(\sup_{t\le k}|w(t)-\tilde{w}(t)|\wedge 1\right)$$

for any sequence $(\alpha_k)\subset(0,\infty)$ with $\sum\alpha_k<\infty$. This metric makes W^d a complete separable metric space (a 'Polish space').

<u>Thm.</u> Under the same assumptions as the previous theorem, for $x\in\mathbb{R}^d$ there exist maps

$$F_x : C(\mathbb{R}_+,\mathbb{R}^m)\longrightarrow C(\mathbb{R}_+,\mathbb{R}^d)$$

measurable w.r.t. the completion of $\mathcal{W}^m$ and w.r.t. $\mathcal{W}^d$ such that

(i) $\forall t \geq 0$, $x \in \mathbb{R}^d$: $F_x(w)_t$ is measurable w.r.t. $\sigma(w(s) : s \leq t)$, for P-a.e. $w \in W^d$.

(ii) $\forall w \in C(\mathbb{R}_+, \mathbb{R}^m)$: $x \mapsto F_x(w) \in C(\mathbb{R}_+, \mathbb{R}^d)$ is continuous.

(iii) $\forall x \in \mathbb{R}^d$, $\forall (\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ satisfying the usual conditions, every $(\mathcal{F}_t)$- Brownian motion B with $B_0 = 0$, the unique solution to $E(\sigma, b)$ with $X_0 = x$ is $X_t = F_x(B)_t$.

(iv) In the set up of (iii), if $U \in \mathcal{F}_0$ then $F_U(B)_t$ is the unique solution with $X_0 = U$.

<u>Proof.</u> For simpler notation, $d=m=1$. Let

$$G_t = \sigma(w(s) : 0 \leq s \leq t) \vee \mathcal{N}, \quad G = G_\infty$$

where $\mathcal{N}$ are the P-nullsets. By the last theorem, there is a unique strong solution X^x to $E(\sigma, b)$ with $X_0^x = x$, with respect to $(W, G, (G_t), P)$ and $B_t(w) = w(t)$.

By the last proposition, choosing (α_k) appropriately,

$$Ed(X^x, X^y)^p \leq \mathbb{E}\Big(\Big(\sum_k \alpha_k \sup_{t \leq k} |X_t^x - X_t^y|\Big)^p\Big)$$

$$\overset{\text{(Jensen)}}{\leq} \sum_k \alpha_k \, \mathbb{E}\Big(\sup_{t \leq k} |X_t^x - X_t^y|^p\Big)$$

$$\leq |x-y|^p \sum_k \alpha_k C_{p,k} \leq C|x-y|^p \qquad (\dagger).$$

A version of Kolmogorov's continuity criterion applies to processes X^x indexed by $\mathbb{R}^d$ and values in a complete metric space, provided (†) holds with $p>d$. Hence there is a modification $(\tilde{X}^x)_{x\in\mathbb{R}^d}$ of $(X^x)_{x\in\mathbb{R}^d}$ that is continuous in $x\in\mathbb{R}^d$. Set

$$F_x(w) = \tilde{X}^x(w) = (\tilde{X}^x_t(w))_{t\geq 0}.$$

Then (ii) is immediate. To see (i), note $w \mapsto X^x_t(w)$ is $\mathcal{G}_t$-measurable. Since $X^x = \tilde{X}^x$ a.s. this gives (i).

To show (iii), assume $(\Omega, \mathcal{F}, \mathbb{P})$ and $\hat{B}$ are given, and set

$$X_t = F_x(\hat{B})_t.$$

Since F_x maps into $C(\mathbb{R}_+, \mathbb{R}^d)$, X is continuous in t. Since $F(\hat{B})$ coincides a.s. with a random variable measurable with respect to the completed filtration generated by $\hat{B}$ and $\mathcal{F}$ contains this filtration, X is adapted. By definition,

$$\tilde{X}_t = x + \int_0^t \sigma(s, \tilde{X}_s)\, dB_s + \int_0^t b(s, \tilde{X}_s)\, ds$$

$$= x + \lim_{m\to\infty} \sum_{i=1}^{n_m} \sigma(t_{i-1}, \tilde{X}_{t^m_{i-1}})(B_{t^m_i} - B_{t^m_{i-1}}) + \int_0^t b(s, \tilde{X}_s)\, ds$$

in $\mathbb{P}$-probability and thus $\mathbb{P}$-a.s. along a subsequence which we now fix

Since $\tilde{X}(w) = F(w)$ thus, for $\mathbb{P}$-a.e. $w \in W^m$,

$$F_x(w)_t = x + \lim_{m\to\infty} \sum_{i=1}^{n_m} \sigma(t_{i-1}, F_x(w)_{t_{i-1}})(F_x(w)_{t_i} - F_x(w)_{t_{i-1}}) + \int_0^t b(s, X_s)\, ds.$$

Since $\hat{B}$ has law P on W^m, substituting $w = \hat{B}$ and then undoing the approximation of the stochastic integral gives

$$X_t = x + \int_0^t \sigma(s, X_s)\, d\hat{B}_s + \int_0^t b(s, X_s)\, ds$$

as claimed.

(iv) omitted (proof is similar).

<u>Cor.</u> The solutions to $E(\sigma, b)$ with $X_0 = x \in \mathbb{R}^d$ can be constructed for all $x \in \mathbb{R}^d$ simultaneously such that a.s. X^x is continuous in the initial condition x.

5.4. Examples of SDEs

<u>The Ornstein-Uhlenbeck process.</u> Let $\lambda > 0$. The Ornstein-Uhlenbeck process is the unique solution to

$$dX_t = -\lambda X_t\, dt + dB_t.$$

It is an example that can be solved explicitly. Apply Itô's formula to $e^{\lambda t} X_t$:

$$d(e^{\lambda t} X_t) = \lambda e^{\lambda t} X_t\, dt + e^{\lambda t}\, dX_t = e^{\lambda t}\, dB_t$$

$$\Leftrightarrow \quad e^{\lambda t} X_t - X_0 = \int_0^t e^{\lambda s}\, dB_s$$

Wiener integral

$$\Leftrightarrow \quad X_t = e^{-\lambda t} X_0 + \int_0^t e^{-\lambda(t-s)}\, dB_s.$$

Exercise. Fix $X_0 = x \in \mathbb{R}$ (or Gaussian). Then (X_t) is a Gaussian process, i.e., $(X_{t_i})_{i=1}^n$ is jointly Gaussian for all $t_0 = 0 < t_1 < \cdots < t_n$.

Fact. $\mathbb{E}X_t = e^{-\lambda t}x$, $\quad \mathrm{Cov}(X_t, X_s) = \frac{1}{2\lambda}\left(e^{-\lambda|t-s|} - e^{-\lambda|t+s|}\right)$

Proof. $\mathbb{E}X_t = e^{-\lambda t}\mathbb{E}X_0 + \underbrace{\mathbb{E}\int_0^t e^{-\lambda(t-s)}\,dB_s}_{0} = e^{-\lambda t}x.$

By the Kunita-Watanabe identity,

$$\mathrm{Cov}(X_t, X_s) = \mathbb{E}\left((X_t - \mathbb{E}X_t)(X_s - \mathbb{E}X_s)\right)$$

$$= \underbrace{\mathbb{E}\left(\int_0^t e^{-\lambda(t-u)}\,dB_u \int_0^s e^{-\lambda(s-u)}\,dB_u\right)}_{\mathbb{E}((H\cdot B)_\infty(K\cdot B)_\infty) = \mathbb{E}\langle H\cdot B, K\cdot B\rangle_\infty = \mathbb{E}(HK\cdot\langle B\rangle)_\infty}$$

$$= \int_0^\infty \left(1_{u<t}\, e^{-\lambda(t-u)}\right)\left(1_{u<s}\, e^{-\lambda(s-u)}\right) du$$

$$= e^{-\lambda(t+s)}\int_0^{s\wedge t} e^{2\lambda u}\,du = \frac{1}{2\lambda}e^{-\lambda(t+s)}\left(e^{2\lambda(s\wedge t)} - 1\right)$$

Cor. $X_t \sim \mathcal{N}\Big(\underbrace{e^{-\lambda t}x}_{\to 0}, \underbrace{\frac{1-e^{-2\lambda t}}{2\lambda}}_{\to \frac{1}{2\lambda}}\Big)$ for every $t \geq 0$, $X_0 = x \in \mathbb{R}$.

as $t \to \infty$

This suggests the distribution $\mathcal{N}(0, \frac{1}{2\lambda})$ is invariant. This is indeed easy to check.

Fact. If $X_0 \sim \mathcal{N}(0, \frac{1}{2\lambda})$ then $X_t \sim \mathcal{N}(0, \frac{1}{2\lambda})$ for all $t \geq 0$, and X_t is a stationary Gaussian process with

$$\mathrm{Cov}(X_t, X_s) = \mathrm{Cov}(X_0, X_{|t-s|}) = \frac{1}{2\lambda}e^{-\lambda|t-s|}.$$

<u>Geometric Brownian motion</u> For $w \in C(\mathbb{R}_+, \mathbb{R})$ define $F_x(w)$ by

$$[F_x(w)](t) = x \exp\left(\sigma w(t) + \left(\mu - \frac{\sigma^2}{2}\right) t\right).$$

If B is a standard Brownian motion with $B_0 = 0$ then $X_t = F_x(B)_t$ satisfies

$$dX_t = \sigma X_t \, dB_t + \mu X_t \, dt.$$

On the other hand, if we choose w to be any <u>smooth</u> path, then $y_t = F_x(w)_t$ satisfies the ODE

$$dy_t = \sigma y_t \, dw_t + \left(\mu - \frac{\sigma^2}{2}\right) dt.$$

Thus the Itô map F satisfies the 'wrong' equation on smooth paths.

<u>5.5. Local Solutions</u>

As for ODEs, solutions to SDEs do not always exist for all times. For SDEs the explosion time is random.

<u>Prop.</u> (Local Itô formula). Let $X = (X^1, \dots, X^d)$ be continuous semimartingales. Let $U \subset \mathbb{R}^d$ be open and let $f : U \to \mathbb{R}^d$ be C^2. Set $T = \inf\{t \geq 0 : X_t \notin U\}$. Then for all $t < T$,

$$f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial f}{\partial x^i}(X_s) \, dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_s) \, d\langle X^i, X^j \rangle_s.$$

Sketch. Apply Itô's formula to X^{T_n} where

$$T_n = \inf\{t \geq 0 : \mathrm{dist}(X_t, U^c) \leq \tfrac{1}{n}\}.$$

Example. Let B be a standard one-dimensional Brownian motion with $B_0 = 1$. Taking $U = (0, \infty)$, $f(x) = \sqrt{x}$ gives

$$\sqrt{B_t} = 1 + \frac{1}{2}\int_0^t B_s^{-1/2}\, ds - \frac{1}{8}\int_0^t B_s^{-3/2}\, ds$$

for $t < T = \inf\{t \geq 0 : B_t = 0\}$.

Thm. Let $U \subset \mathbb{R}^d$ be open and $b : \mathbb{R}_+ \times U \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times U \to \mathbb{R}^{d \times m}$ be locally Lipschitz continuous. Then for every $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and B, every $x \in U$, there exists a stopping time T such that, for $t < T$,

$$X_t = x + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s,$$

and for all compact $K \subset U$, $\sup\{t < T : X_t \in K\} < T$. This stopping time T is called the explosion time.

Proof. Fix $K_n \subset U$ compact such that $K_{n+1} \supseteq K_n$ and $\bigcup_n K_n = U$. One can find b_n and σ_n defined on all of $\mathbb{R}^d$ such that $b_n|_{\mathbb{R}_+ \times K_n} = b|_{\mathbb{R}_+ \times K_n}$ and $\sigma_n|_{\mathbb{R}_+ \times K_n} = \sigma|_{\mathbb{R}_+ \times K_n}$ and such that b_n and σ_n are globally Lipschitz continuous. Hence, for any $x \in U$ and n large enough that $x \in K_n$, there are unique solutions X^n to $E(\sigma_n, b_n)$ with $X_0 = x$. Let

$$T_n = \inf\{t \geq 0 : X_t^n \notin K_n\}.$$

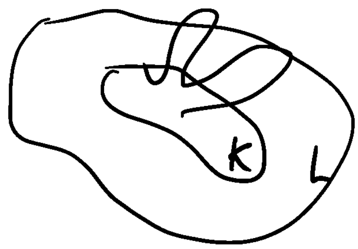
By uniqueness, X^{n+1} also solves $E(\sigma_n, b_n)$ up to time T_n. Thus $X^{n+1}_t = X^n_t$ for $t < T_n$ and we can define X_t for $t < T := \sup_n T_n$ by requiring that $X_t = X^n_t$ for $t < T_n$.

Claim: Let $K \subset U$ be compact. Then on $\{T < \infty\}$ a.s.

$$\sup\{t < T : X_t \in K\} < T.$$

Let L be another compact set such that $K \subset \mathring{L} \subset L \subset U$. Let $\varphi : U \to \mathbb{R}$ be C^∞ such that $\varphi|_K = 1$ and $\varphi|_{\mathring{L}^c} = 0$.

Let
$$R_1 = \inf\{t \geq 0 : X_t \notin L\}$$
$$S_n = \inf\{t \geq R_n : X_t \in K\} \wedge T$$
$$R_n = \inf\{t \geq S_{n-1} : X_t \notin L\}$$



Let N be the number of crossings of X from $\mathring{L}^c$ to K.

Then on $\{T \leq t, N \geq n\}$,

$$n = \sum_{k=1}^{n} (\varphi(X_{S_k}) - \varphi(X_{R_k}))$$
$$= \sum_{k=1}^{n} \int_{R_k}^{S_k} \left(\nabla \varphi(X_s) \cdot dX_s + \frac{1}{2} \sum_{i=1}^{d} \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(X_s) \, d\langle X^i, X^j \rangle_s \right)$$
$$= \int_0^t \left(H^n_s \, dB_s + \tilde{H}^n_s \, ds \right) = Z^n_t$$

with H^n and $\tilde{H}^n$ predictable and bounded uniformly in n.

$\Rightarrow n^2 \mathbb{1}_{\{T \leq t, N \geq n\}} \leq (Z^n_t)^2 \Rightarrow \mathbb{P}(T \leq t, N \geq n) \leq n^{-2} \mathbb{E}((Z^n_t)^2) \leq \frac{C(t)}{n^2}$.

$\Rightarrow \mathbb{P}(T \leq t, N = \infty) = 0 \Rightarrow \mathbb{P}(T < \infty, N = \infty) = 0$ as claimed.

<u>Example.</u> Consider the SDEs

$$dX_t^i = b^i(X_t)\,dt + dB_t^i, \quad X_0 = x.$$

Assume there are $a>0$, $c>0$ such that

$$x \cdot b(x) \le a|x|^2 + c.$$

Then the SDE has a global solution, i.e., $T=\infty$ a.s.

<u>Proof.</u> Let $T_n = \inf\{t\ge 0 : |X_t|^2 \ge n\}$. Then by Itô's formula

$$\begin{aligned}
\mathbb{E}|X_{t\wedge T_n}|^2 &= \mathbb{E}|X_0|^2 + 2\,\mathbb{E}\int_0^{t\wedge T_n} X_s \cdot b(X_s)\,ds + (t\wedge T_n)d \\
&\le \mathbb{E}|X_0|^2 + 2a\,\mathbb{E}\int_0^{t\wedge T_n} |X_s|^2\,ds + (d+c)t \\
&\le \mathbb{E}|X_0|^2 + (d+c)t + 2a\int_0^t \mathbb{E}|X_{s\wedge T_n}|^2\,ds.
\end{aligned}$$

By Gronwall's inequality,

$$\mathbb{E}|X_{t\wedge T_n}|^2 \le \underbrace{(\mathbb{E}|X_0|^2 + (d+c)t)\,e^{2at}}_{C(t)}$$

$$\Rightarrow \mathbb{P}(T_n \le t) \le \frac{C(t)}{n}$$

The explosion time T satisfies $T = \lim_{n\to\infty} T_n$ by the theorem.
Hence by Fatou,

$$\mathbb{P}(T\le t) = 0 \quad \text{for every } t>0.$$

$$\Rightarrow \mathbb{P}(T=\infty) = 1.$$

6. Applications to PDEs and Markov processes

6.1. Dirichlet-Poisson problem

Let $U \subset \mathbb{R}^d$ be open, $U \neq \emptyset$. For locally bounded coefficients $a: \bar{U} \to \mathbb{R}^{d\times d}_{sym}$ and $b: \bar{U} \to \mathbb{R}^d$, consider

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x^i}(x)$$

<u>Defn.</u> L is <u>uniformly elliptic</u> if there is $\lambda > 0$ such that

$$\xi^T a(x)\xi \geq \lambda |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d.$$

<u>Dirichlet-Poisson problem.</u> Given $f \in C(\bar{U})$, $g \in C(\partial U)$, find $u \in C^2(\bar{U}) = C^2(U) \cap C(\bar{U})$ such that

$$\begin{cases} -Lu(x) = f(x) & \text{for } x \in U \\ u(x) = g(x) & \text{for } x \in \partial U \end{cases}$$

Dirichlet problem: $f = 0$
Poisson problem: $g = 0$

<u>Thm</u> ($\rightarrow$ Evans, Gilbarg-Trudinger). Assume U is bounded with smooth boundary, that a and b are Hölder continuous and that L is uniformly elliptic. Then for every Hölder continuous $f: \bar{U} \to \mathbb{R}$ and every continuous $g: \partial U \to \mathbb{R}$, the Dirichlet-Poisson problem has a solution.

Thm. Let $U \subset \mathbb{R}^d$ be open, bounded, $U \neq \emptyset$, let $b : \bar{U} \to \mathbb{R}^d$ and $\sigma : \bar{U} \to \mathbb{R}^{d \times m}$ be bounded Borel functions, and assume that $a = \sigma\sigma^T : \bar{U} \to \mathbb{R}^{d \times d}$ is uniformly elliptic. Assume $u \in C^2(\bar{U})$ is a solution to (DP) and that X is a solution to $E(\sigma, b)$ with $X_0 = x$. Let $T_U = \inf\{ t \geq 0 : X_t \notin U\}$. Then $\mathbb{E}\, T_U < \infty$ and

$$u(x) = \mathbb{E}_x\left(u(X_{T_U}) - \int_0^{T_U} Lu(X_s)\, ds\right)$$

$$= \mathbb{E}_x\left(g(X_{T_U}) + \int_0^{T_U} f(X_s)\, ds\right) \quad \text{(Dynkin's formula)}$$

law under which $X_0 = x$

Exercise. Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be bounded Borel functions and assume X is a solution to $E(\sigma, b)$. Then for $f : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ that is C^1 in t and C^2 in x,

$$M_t^f = f(t, X_t) - f(0, X_0) - \int_0^t \left(\frac{\partial}{\partial s} + L\right) f(s, X_s)\, ds$$

is a continuous local martingale.

Defn. L is called the generator of X.

Example. • $dX = dB \rightarrow L = \frac{1}{2}\Delta$

• $dX = -X\,dt + dB$ (Ornstein-Uhlenbeck process)
$\rightarrow L = \frac{1}{2}\Delta - X \cdot \nabla$

(*) below: For a, b Hölder, ∂U smooth, v exists by above general theorem. But in general (exercise) one can find such a v explicitly.
(Hint: set $v(x) = C e^{\alpha R - \alpha x^1}$ for constants α, R and C large.)

<u>Proof</u> (theorem). Let $T_n = \inf\{t \geq 0 : X_t \notin U_n\}$ where

$$U_n = \{x \in U : \mathrm{dist}(x, \partial U) > \tfrac{1}{n}\}.$$

Then there are $u_n \in C_b^2(\mathbb{R}^d)$ s.t. $u_n|_{U_n} = u|_{U_n}$. Note that

$$M_t^n = (M^{u_n})_t^{T_n} = u_n(X_{t\wedge T_n}) - u_n(X_0) - \int_0^{t\wedge T_n} Lu_n(X_s)\, ds$$

is a local martingale, bounded for $t \leq t_0$ for any $t_0 > 0$, so a (true) martingale. Thus for n large enough,

$$\underset{\substack{\uparrow \\ n \text{ large depending} \\ \text{on } x \in U}}{u(x) = u_n(x)} = \mathbb{E}\Big(\underbrace{u_n(X_{t\wedge T_n})}_{u(X_{t\wedge T_n})} - \int_0^{t\wedge T_n} \underbrace{Lu_n(X_s)}_{-f(X_s)}\, ds \Big)$$

To take the limit $t \wedge T_n \to T_U$, we will need $\mathbb{E}T_U < \infty$.

To see $\mathbb{E}T_U < \infty$, let v be a solution$^{(*)}$ to (DP) with $f(x) \geq 1$ $\forall x$ and $g(x) \geq 0$ $\forall x$. Then

$$\mathbb{E}(t \wedge T_n) \geq \mathbb{E}\Big(\int_0^{t\wedge T_n} \underbrace{(-Lv)(X_s)}_{=1}\, ds \Big) = \underbrace{v(x) - \mathbb{E}(v(X_{t\wedge T_n}))}_{\leq 2\|v\|_\infty}.$$

By monotone convergence and since $T_n \uparrow T_U$ a.s., thus

$$\mathbb{E}T_U \leq 2\|v\|_\infty.$$

Claim: $u(x) = \mathbb{E}\Big(u(X_{T_U}) - \int_0^{T_U} f(X_s)\, ds\Big)$.

Indeed, since $t \wedge T_n \uparrow T_U$ as $n \to \infty$, $t \to \infty$, and since

$$\mathbb{E}\Big(\int_0^{T_u} |f(X_s)|\, ds\Big) \le \|f\|_\infty\, \mathbb{E}T_u < \infty,$$

the DCT implies

$$\mathbb{E}\Big(\int_0^{t\wedge T_n} f(X_s)\, ds\Big) \to \mathbb{E}\Big(\int_0^{T_u} f(X_s)\, ds\Big).$$

Since u is continuous on $\bar{\mathcal{U}}$, also by DCT

$$\mathbb{E}(u(X_{t\wedge T_n})) \to \mathbb{E}(u(X_{T_u})).$$

6.2. Cauchy problem

The Cauchy problem is to find, given $f \in C_b^2(\mathbb{R}^d)$, a bounded solution $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ that is C^1 in t and C^2 in x, to

$$(\mathrm{CP}) \begin{cases} \frac{\partial u}{\partial t} = Lu & \text{on } (0,\infty)\times\mathbb{R}^d \\ u(0,\cdot) = f & \text{on } \mathbb{R}^d \end{cases}$$

Thm. ($\to$ Evans, Gilbarg-Trudinger, ...). Assume L is uniformly elliptic. Then for every $f \in C_b^2(\mathbb{R}^d)$, there is a solution to (CP).

Thm. Let u be a bounded solution to (CP). Then for any solution X to $E(\sigma, b)$ with $X_0 = x$, $x \in \mathbb{R}^d$, $0 \le s < t$,

$$u(t,x) = \mathbb{E}_x f(X_t)$$

and, more generally,

$$\mathbb{E}_x(f(X_t) \mid \mathcal{F}_s) = u(t-s, X_s).$$

Proof. By assumption $g(s,x) = u(t-s,x)$ satisfies

$$\left(\frac{\partial}{\partial s} + L\right) g(s,x) = 0.$$

Therefore $g(s,X_s) - g(0,x)$ is a martingale. Thus

$$u(t-s, X_s) = g(s,X_s) = \mathbb{E}(g(t,X_t) \mid \mathcal{F}_s) = \mathbb{E}(f(X_s) \mid \mathcal{F}_s).$$

Thm (Feynman-Kac formula). Let $f \in C_b^2(\mathbb{R}^d)$, $V \in C_b(\mathbb{R}^d)$ and suppose $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is a bounded solution to

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu + Vu & \text{on } \mathbb{R}_+ \times \mathbb{R}^d \\ u(0,\cdot) = f \end{cases}$$

pointwise multiplication

Let X be a solution to $E(\sigma, b)$ with $X_0 = x \in \mathbb{R}^d$. Then for all $t \geq 0$,

$$u(t,x) = \mathbb{E}_x\left(f(X_t) \exp\left(\int_0^t V(X_s)\, ds\right)\right).$$

Proof. Let $E_t = \exp\left(\int_0^t V(X_s)\, ds\right)$. For $s<t$, set

$$M_s = u(t-s, X_s) E_s.$$

Then M is a martingale on $[0,t]$. Indeed, it is bounded and

$$\begin{aligned} dM_s &= -\frac{\partial u}{\partial t}(t-s, X_s) E_s\, ds + \nabla u(t-s, X_s) E_s\, \sigma(X_s)\, dB_s \\ &\quad + Lu(t-s, X_s) E_s\, ds + u(t-s, X_s) V(X_s) E_s\, ds \\ &= 0 + \nabla u(t-s, X_s) E_s\, \sigma(X_s)\, dB_s. \end{aligned}$$

Hence

$$u(t,x) = M_0 = \mathbb{E} M_t = \mathbb{E} u(0, X_t) E_t = \mathbb{E} f(X_t) E_t.$$

6.3. Markov property

Let $B(\mathbb{R}^d)$ be the Banach space of bounded Borel functions with $\|f\| = \sup_{x \in \mathbb{R}^d} |f(x)|$ for $f \in B(\mathbb{R}^d)$.

Defn. (i) A collection of bounded linear operators Q_t on $B(\mathbb{R}^d)$ is a transition semigroup if $Q_t f \geq 0$ a.e. if $f \geq 0$ a.e., $Q_t 1 = 1$ where $1(x) = 1$ for all $x \in \mathbb{R}^d$, $\|Q_t\| \leq 1$, and

$$Q_{t+s} = Q_t Q_s \quad \text{for all } t,s \geq 0.$$

(ii) An $(\mathcal{F}_t)$-adapted process X is a Markov process with transition semigroup (Q_t) if

$$\mathbb{E}(f(X_{s+t}) \mid \mathcal{F}_s) = Q_t f(X_s) \quad \text{for all } s,t \geq 0,\ f \in B(\mathbb{R}^d).$$

Thm. Let $b: \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma: \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be Lipschitz. Assume X is a solution to $E(\sigma, b)$ on some $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. Then X is a Markov process with semigroup

$$Q_t f(x) = \mathbb{E}_x f(X_t) = \int f(F_x(w)_t)\, P(dw)$$

where F_x is the solution map on Wiener space and P is the Wiener measure.

Proof. Let X be a solution to $E(\sigma, b)$.

Claim: $\mathbb{E}(f(X_{t+s}) \mid \mathcal{F}_s) = Q_t f(X_s)$

Indeed, $X_t = X_0 + \int_0^t \sigma(X_u)\, dB_u + \int_0^t b(X_u)\, du$

$$X_{t+s} = X_0 + \int_0^{t+s} \sigma(X_u)\, dB_u + \int_0^{t+s} b(X_u)\, du$$

Setting $\tilde{X}_t = X_{t+s}$, $\tilde{\mathcal{F}}_t = \mathcal{F}_{t+s}$, and $\tilde{B}_t = B_{t+s} - B_s$, note that $(\Omega, \mathcal{F}, (\tilde{\mathcal{F}}_t), \mathbb{P})$ is another filtered probability space obeying the usual conditions and $\tilde{B}$ an $(\tilde{\mathcal{F}}_t)$-Brownian motion.

We have

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t \sigma(\tilde{X}_u)\, d\tilde{B}_u + \int_0^t b(\tilde{X}_u)\, du$$

Indeed, this follows from $\int_s^{s+t} \sigma(X_u)\, dB_u = \int_0^t \sigma(\tilde{X}_u)\, d\tilde{B}_u$ which can be seen by approximating both sides by sums.

Thus $\tilde{X}$ solves $E(\sigma, b)$ with $\tilde{X}_0 = X_s$ and therefore in terms of the solution map $\tilde{X} = F_{X_s}(\tilde{B})$ a.s.

$$\Rightarrow \mathbb{E}(f(X_{s+t}) \mid \mathcal{F}_s) = \mathbb{E}(f(\tilde{X}_t) \mid \mathcal{F}_s) \underset{\substack{\uparrow \\ \tilde{B} \text{ is indep. of } X_s}}{=} \mathbb{E}(f(F_{X_s}(\tilde{B})_t \mid \mathcal{F}_s) = Q_t f(X_s)$$

and

$$Q_{t+s} f(x) = \mathbb{E}_x f(X_{t+s}) = \mathbb{E}_x\big(\underbrace{\mathbb{E}_x(f(X_{t+s}) \mid \mathcal{F}_s)}_{Q_t f(X_s)}\big) = Q_s Q_t f(x).$$

Rk. Using the strong Markov property for Brownian motion, one can prove this property for X similarly.

Let $C_0(\mathbb{R}^d)$ be the Banach space of continuous functions tending to 0 at ∞ with norm $\|f\| = \sup_{x\in\mathbb{R}^d}|f(x)|$.

<u>Defn.</u> A transition semigroup has the <u>Feller property</u> if

(i) $\forall f\in C_0, t>0: \ Q_t f \in C_0$

(ii) $\forall f\in C_0: \|Q_t f - f\| \to 0$ as $t\to 0$.

<u>Thm.</u> The semigroup of the last theorem is Feller.

<u>Proof</u> (assuming σ and b are bounded). Let $f\in C_0$.

Claim: $Q_t f$ is continuous

Indeed, since $x\mapsto F_x(w)$ is continuous, by DCT,

$$|Q_t f(x) - Q_t f(y)| \leq \int |f(F_x(w)_t) - f(F_y(w)_t)| \, P(dw) \to 0.$$

Claim: $Q_t f(x) \to 0$ as $|x|\to 0$

Since $X_t^x = x + \int_0^t b(X_s^x)\,ds + \int_0^t \sigma(X_s^x)\,dB_s$ and σ, b are bounded,

$$\mathbb{E}|X_t^x - x|^2 \leq C(t^2+t)$$

$$\overset{\text{Markov}}{\Rightarrow} \ \mathbb{P}(|X_t^x - x| > A) \leq A^{-1}C_t \to 0 \text{ as } A\to\infty.$$

$$\Rightarrow |Q_t f(x)| \leq \mathbb{E}|f(X_t^x)| \mathbb{1}_{|X_t^x - x|\leq A} + \|f\| \, \mathbb{P}(|X_t^x - x| > A)$$

$$\Rightarrow \limsup_{|x|\to\infty} |Q_t f(x)| \leq 0 + \|f\| \, \mathbb{P}(|X_t^x - x| > A) \quad \forall A$$

since $f(x)\to 0$ as $x\to\infty$.

$$\overset{A\to\infty}{\Rightarrow} \lim_{|x|\to\infty} |Q_t f(x)| = 0.$$

Claim: $\|Q_t f - f\| \to 0$

$$\sup_x |\mathbb{E}(f(X_t^x) - f(x))| \le \sup_{|x-y|\le\varepsilon} |f(x)-f(y)| + 2\|f\| \underbrace{\sup_x \mathbb{P}(|X_t^x - x| > \varepsilon)}_{\substack{\le C\varepsilon^{-1}(t^2+t) \\ \to 0 \text{ as } t\to 0}}$$

$$\Rightarrow \lim_{t\to 0} \|Q_t f - f\| = 0.$$

<u>Defn.</u> Let

$$D(L) = \{ f\in C_0 : \tfrac{1}{t}(Q_t f - f) \text{ converges in } C_0 \text{ as } t\downarrow 0 \}$$

$$Lf = \lim_{t\downarrow 0} \tfrac{1}{t}(Q_t f - f) \quad \text{for } f\in D(L)$$

The linear operator $L: D(L) \to C_0$ is called the <u>generator</u> of the semigroup (Q_t).

<u>Prop.</u> Let $f\in D(L)$. Then $Q_t f\in D(L)$, $L(Q_t f) = Q_t(Lf)$, and

$$Q_t f = f + \int_0^t Q_s Lf \, ds = f + \int_0^t LQ_s f \, ds$$

<u>Proof.</u> To see $Q_t f \in D(L)$ and $L(Q_t f) = Q_t(Lf)$, note

$$\frac{1}{s}\Big(Q_s(Q_t f) - Q_t f\Big) = Q_t\Big(\underbrace{\tfrac{1}{s}(Q_s f - f)}_{\to Lf}\Big) \to Q_t(Lf) \text{ as } s\downarrow 0.$$

since $\|Q_t\| \le 1$.

Since the left-hand side is $\frac{1}{s}(Q_{t+s} f - Q_t f)$, $Q_t f$ is also right-differentiable with right-derivative $Q_t Lf$. Since this right-derivative is continuous in t, in fact $Q_t f$ is differentiable in t.

<u>Thm</u> Let $f, g \in C_0$. Then the following are equivalent:

(i) $f \in D(L)$ and $Lf = g$.

(ii) For every $x \in \mathbb{R}^d$, $f(X_t^x) - \int_0^t g(X_s^x)\, ds$ is a martingale.

<u>Proof.</u> (i) $\Rightarrow$ (ii).

$$\mathbb{E}\Big(f(X_{t+s}^x) - \int_0^{t+s} g(X_r^x)\, dr \,\Big|\, \mathcal{F}_t \Big)$$

$$= \mathbb{E}\Big(\underbrace{Q_s f(X_t^x)} - \int_0^{t+s} g(X_r^x)\, dr \,\Big|\, \mathcal{F}_t \Big)$$

$$Q_s f = f + \int_0^s Q_r g \, dr \quad \text{by (i)}$$

$$= f(X_t^x) + \int_0^t g(X_r^x)\, dr + \underbrace{\mathbb{E}\Big(\int_0^s Q_r g(X_t^x)\, dr - \int_0^s g(X_{r+t}^x)\, dr \,\Big|\, \mathcal{F}_t \Big)}_{=0}$$

(ii) $\Rightarrow$ (i). By (ii),

$$f(x) = \mathbb{E}\Big(f(X_t^x) - \int_0^t g(X_r^x)\, dr \Big)$$

$$= Q_t f(x) - \int_0^t Q_r g(x)\, dr$$

$$\Rightarrow \frac{1}{t}(Q_t f - f) = \frac{1}{t}\int_0^t Q_r g(x)\, dr \longrightarrow g(x) \quad \text{as } t \downarrow 0.$$

<u>Cor.</u> The generator of the semigroup of the SDE as in the last theorems satisfies $C_c^2(\mathbb{R}^d) \subset D(L)$,

$\forall f \in C_c^2(\mathbb{R}^d)$, Lf is as in Section 6.1.

6.4. Convergence to equilibrium

<u>Defn.</u> Let (Q_t) be a transition semigroup and let μ be a probability measure on $\mathbb{R}^d$. Then μ is

(i) <u>invariant</u> under (Q_t) if

$$\int Q_t f(x)\,\mu(dx) = \int f(x)\,\mu(dx) \quad \forall f \in B(\mathbb{R}^d)$$

(ii) <u>reversible</u> under (Q_t) if

$$\int g(x)\, Q_t f(x)\,\mu(dx) = \int Q_t g(x)\, f(x)\,\mu(dx) \quad \forall f,g \in B(\mathbb{R}^d).$$

<u>Fact.</u> reversible $\Rightarrow$ invariant

<u>Proof.</u> Take $g=1$ and use $Q_t 1 = 1$.

<u>Fact.</u> Under the same assumptions,

(i) invariant iff $\int Lf\, d\mu = 0$ for all $f \in D(L)$

(ii) reversible iff $\int g Lf\, d\mu = \int (Lg) f\, d\mu$ for all $f,g \in D(L)$.

<u>Rk.</u> In practice, $D(L)$ is rarely known explicitly. However, if $D \subset D(L)$ is a dense subspace such that $Q_t f \in D$ for $f \in D$ and $t>0$ then it suffices to check the conditions for $f \in D$. For example, for $f \in D$,

$$\frac{d}{dt}\int Q_t g\, d\mu = \int L\overbrace{Q_t g}^{f}\, d\mu = \int Lf\, d\mu = 0.$$

Prop. Let Q_t be the transition semigroup of the SDE

$$dX_t = -\nabla H(X_t)\,dt + \sqrt{2}\,dB_t.$$

Assume that $H \in C_b^2(\mathbb{R}^d)$ with $\int e^{-H}(1+|\nabla H|) < \infty$. Then the probability measure $\mu(dx) = \frac{1}{Z} e^{-H(x)}\,dx$ is reversible.

Proof. For $f, g \in C_b^2(\mathbb{R}^d)$,

$$\int g(Lf)e^{-H}\,dx = \int g\underbrace{(\Delta f - \nabla H\cdot\nabla f)\,e^{-H}}_{\nabla\cdot((\nabla f)e^{-H})}dx$$

$$= -\int \nabla f\cdot\nabla g\, e^{-H}dx \overset{\text{same argument}}{=} \int (Lf)g\,e^{-H}\,dx.$$

Thm. Assume that H is uniformly convex, i.e., there is $\lambda > 0$ such that

$$\xi^T \operatorname{Hess} H(x)\,\xi \geq \lambda|\xi|^2 \quad \text{for all } x\in\mathbb{R}^d,\ \xi\in\mathbb{R}^d.$$

Let X and Y be two solutions to the SDE w.r.t. the same Brownian motion. Then, a.s.,

$$|X_t - Y_t| \leq |X_0 - Y_0|\, e^{-\lambda t}$$

indep. of dimension

Proof. By Itô's formula,

$$e^{2\lambda t}|X_t - Y_t|^2 = |X_0 - Y_0|^2 + \int_0^t 2\lambda\,|X_s - Y_s|^2 e^{2\lambda s}\,ds$$

$$-2\int_0^t \underbrace{(X_s - Y_s)\cdot(\nabla H(X_s) - \nabla H(Y_s))}_{\geq \lambda|X_s - Y_s|^2}\, e^{2\lambda s}\,ds$$

$$\leq |X_0 - Y_0|^2.$$

Cor. For every $f\in C_b(\mathbb{R}^d)$, $\mathbb{E}_x f(X_t) \to \mathbb{E}_\mu f$ where μ is the invariant probability measure $\mu(dx) \propto e^{-H(x)}dx$.

Proof. It suffices to assume that $f\in C_b^1$. Then if $Y_0\sim\mu$,

$$\mathbb{E}f(X_t) = \mathbb{E}f(Y_t) + O(\|\nabla f\|_\infty \underbrace{\mathbb{E}|X_t - Y_t|}_{\le e^{-\lambda t}\mathbb{E}|X_0 - Y_0|}) \to \mathbb{E}f(Y_0) = \mathbb{E}_\mu f.$$

Cor. The measure μ satisfies the Poincaré inequality

$$\mathrm{Var}_\mu f \le \frac{1}{\lambda}\mathbb{E}_\mu|\nabla f|^2 \qquad \forall f\in C_b^1(\mathbb{R}^d).$$

Proof. By the mean-value theorem,

$$|Q_tf(x) - Q_tf(y)| = \mathbb{E}|f(X_t^x) - f(X_t^y)| \le e^{-\lambda t}\,\mathbb{E}|\nabla f(Z_t)|\cdot|x-y|$$

where Z_t lies on the line from X_t^x to X_t^y.

$$\Rightarrow \frac{|Q_tf(x) - Q_tf(y)|}{|x-y|} \le e^{-\lambda t}\,\mathbb{E}|\nabla f(Z_t)|$$

By continuity of the solution map,

$$\Rightarrow |\nabla Q_tf(x)| \le e^{-\lambda t}\,\mathbb{E}|\nabla f(X_t)| = e^{-\lambda t} Q_t|\nabla f|(x).$$

Thus

$$\begin{aligned}\mathrm{Var}_\mu f = \mathbb{E}_\mu f^2 - (\mathbb{E}_\mu f)^2 &= \lim_{t\to\infty}\left(\mathbb{E}_\mu(Q_0f)^2 - \mathbb{E}_\mu(Q_tf)^2\right)\\ &= -\int_0^\infty \mathbb{E}_\mu\, 2(Q_sf)L(Q_sf)\,ds\\ &= 2\int_0^\infty \mathbb{E}_\mu|\nabla Q_tf|^2\\ &\le 2\int_0^\infty e^{-2\lambda t}\,\mathbb{E}_\mu|\nabla f|^2 = \frac{1}{\lambda}\mathbb{E}_\mu|\nabla f|^2.\end{aligned}$$

In fact, the following stronger inequality holds.

<u>Cor.</u> The measure μ satisfies the <u>Log-Sobolev inequality</u>:
For all $f \in C_b^1(\mathbb{R}^d)$ with $f > 0$,

$$\underbrace{\mathbb{E}_\mu(f \log f) - (\mathbb{E}_\mu f)(\log \mathbb{E}_\mu f)}_{=:\, \mathrm{Ent}_\mu(f)} \le \frac{1}{2\lambda} \mathbb{E}_\mu \frac{|\nabla f|^2}{f}$$

<u>Proof.</u> As in the last proof,

$$|\nabla Q_t f| \le e^{-\lambda t} Q_t |\nabla f|.$$

Since $Q_t f(x) = \mathbb{E} f(X_t^x)$, for all strictly positive f,

$$\frac{(Q_t |\nabla f|)^2}{Q_t f} = \frac{(\mathbb{E} |\nabla f|)^2}{\mathbb{E} f} \quad \text{if } \mathbb{E} \text{ denotes the expectation w.r.t. the law of } X_t^x$$

$$= \left(\mathbb{E}_f \frac{|\nabla f|}{f}\right)^2 (\mathbb{E} f) \quad \text{if } \mathbb{E}_f X := \mathbb{E}(fX)/\mathbb{E} f$$

$$\overset{(CS)}{\le} \left(\mathbb{E}_f \frac{|\nabla f|^2}{f^2}\right)(\mathbb{E} f) = \mathbb{E} \frac{|\nabla f|^2}{f} = Q_t\left(\frac{|\nabla f|^2}{f}\right).$$

$$\Rightarrow \frac{|\nabla Q_t f|^2}{Q_t f} \le e^{-2\lambda t} Q_t\left(\frac{|\nabla f|^2}{f}\right).$$

$$\Rightarrow \mathbb{E}_\mu f \log f - \mathbb{E}_\mu f (\log \mathbb{E}_\mu f) = \int_0^\infty \mathbb{E}_\mu \frac{|\nabla Q_t f|^2}{Q_t f} \le \frac{1}{2\lambda} \mathbb{E}_\mu \frac{|\nabla f|^2}{f}.$$

<u>Exercise.</u> Poincaré ineq. $\Leftrightarrow \mathrm{Var}_\mu(Q_t f) \le e^{-2\lambda t} \mathrm{Var}_\mu f$

Log-Sob. ineq. $\Leftrightarrow \mathrm{Ent}_\mu(Q_t f) \le e^{-2\lambda t} \mathrm{Ent}_\mu(f)$.

Log-Sob. ineq. $\Rightarrow$ Poincaré ineq.