Roland Bauerschmidt ⟨rb812@cam.ac.uk⟩, *Daniel Heydecker* ⟨dh489@cam.ac.uk⟩ *Lent 2020*

Problems marked with (†) may be handed in for marking (CCA pidgeonhole G/H). Problems marked with (⋆) are additional questions

Problem 1. Suppose that $(Z_t)_{t\geq 0}$ is a continuous local martingale which is strictly positive almost surely. Show that there is a unique continuous local martingale M such that $Z = \mathcal{E}(M)$, where

$$\mathcal{E}(M)_t = \exp(M_t - \tfrac{1}{2}\langle M\rangle_t).$$

Problem 2. Let M be a continuous local martingale with $M_0 = 0$. For any $a, b > 0$, show that

$$\mathbb{P}\left(\sup_{t\geq 0} M_t \geq a, \langle M\rangle_\infty \leq b\right) \leq \exp\left(-\frac{a^2}{2b}\right).$$

Problem 3. (†) Let B be a standard Brownian motion and, for $a, b > 0$, let $\tau_{a,b} = \inf\{t \geq 0 : B_t + bt = a\}$. Use Girsanov's theorem to prove that the density of $\tau_{a,b}$ is given by

$$a(2\pi t^3)^{-1/2}\exp(-(a-bt)^2/2t).$$

Problem 4. Suppose that M is a continuous local martingale with $\langle M\rangle_t \to \infty$ almost surely as $t \to \infty$. Show that $M_t/\langle M\rangle_t \to 0$ as $t \to \infty$ and conclude that $\mathcal{E}(M)_t \to 0$ almost surely.

Problem 5. (Gronwall's lemma) Let $T > 0$ and let f be a non-negative, bounded, measurable function on $[0, T]$. Suppose that there exist $a, b \geq 0$ such that

$$f(t) \leq a + b\int_0^t f(s)ds \quad \text{for all} \quad t \in [0, T].$$

Show that $f(t) \leq ae^{bt}$ for all $t \in [0, T]$.

Problem 6. (†) Suppose that X is a continuous local martingale with quadratic variation

$$\langle X\rangle_t = \int_0^t A_s ds$$

for a non-negative, previsible process $(A_t)_{t\geq 0}$. Show that there exists a Brownian motion B (possibly defined on a larger probability space) such that

$$X_t = \int_0^t A_s^{1/2} dB_s.$$

Problem 7. Suppose that σ and b are Lipschitz. Explain why uniqueness in law holds for the SDE $dX_t = \sigma(X_t)dB_t + b(X_t)dt$.

Problem 8. (†) Suppose that σ, b and σ_n, b_n for $n \in \mathbb{N}$ are Lipschitz with constant K uniformly in n. Suppose that $\sigma_n \to \sigma$ and $b_n \to b$ uniformly. Suppose that X and X^n are defined by

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad X_0 = x \tag{1}$$

$$dX_t^n = \sigma_n(X_t^n)dB_t + b_n(X_t^n)dt, \quad X_0^n = x. \tag{2}$$

Show for each $t > 0$ that

$$\mathbb{E}\left(\sup_{0\leq s\leq t} |X_s^n - X_s|^2\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Suppose instead that b_n, σ_n are only assumed to be *continuous*, and b, σ are Lipschitz. Suppose that X^n, X still satisfy (1-2), although this may not uniquely determine X^n. What happens now?

Problem 9. Let b be bounded and σ be bounded and continuous.

i. Suppose that X is a weak solution of the SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$. Show that the process

$$f(X_t) - \int_0^t \left(b(X_s)f'(X_s) - \frac{1}{2}\sigma^2(X_s)f''(X_s) \right) ds$$

is a local martingale for all $f \in C^2$.

ii. Let X be a continuous, adapted process such that

$$f(X_t) - \int_0^t \left(b(X_s)f'(X_s) - \frac{1}{2}\sigma^2(X_s)f''(X_s) \right) ds$$

is a local martingale for each $f \in C^2$. Suppose that $\sigma(x) > 0$ for all x. Using Problem 6, show that there exists a Brownian motion such that $dX_t = b(X_t)dt + \sigma(X_t)dW_t$.

Problem 10. (The Reflection Principle Revisited) Using the results of this course, give a *short* proof of the reflection principle: if B is a standard Brownian motion relative to a filtration $(\mathcal{F}_t)_{t\geq 0}$, and T is a stopping time for the same filtration, then

$$W_t = \begin{cases} B_t & t \leq T; \\ 2B_T - B_t & t > T. \end{cases}$$

is also a standard Brownian Motion.

Problem 11. (Brownian Bridges) Let W be a standard Brownian motion.

i. Let $B_t = W_t - tW_1$. Show that $(B_t)_{t\in[0,1]}$ is a continuous, mean-zero Gaussian process. What is the covariance $\mathbb{E}[B_s B_t]$?

ii. Is B adapted to the filtration generated by W?

iii. Let

$$dX_t = -\frac{X_t}{1-t}dt + dW_t, \quad X_0 = 0.$$

Verify that

$$X_t = (1-t)\int_0^t \frac{dW_s}{1-s} \quad \text{for} \quad 0 \leq t < 1.$$

Show that $X_t \to 0$ as $t \uparrow 1$.

iv. Show that X is a continuous, mean-zero Gaussian process with the same covariance as B, which we call a *Brownian bridge*.

v (⋆). For $y \in \mathbb{R}$, define a process B^y by $B^y_t = B_t + ty = W_t + t(y - W_1)$, $0 \leq t \leq 1$. Let $F : C[0,1] \to \mathbb{R}$ be bounded, and continuous with respect to the uniform norm, and define

$$f(y) = \mathbb{E}\left[F(B^y)\right]. \tag{3}$$

Show that f is bounded and continuous, and that $\mathbb{E}(F(W)|W_1) = f(W_1)$ almost surely.

vi (⋆). For $\epsilon > 0$, let us write μ_ϵ for the probability measure on $(C[0,1], \mathcal{W})$ given by

$$\mu_\epsilon(A) = \frac{\mathbb{P}(W \in A \text{ and } |W_1| < \epsilon)}{\mathbb{P}(|W_1| < \epsilon)} \tag{4}$$

and let μ_0 be the law of B. Use the previous part to show that $\mu_\epsilon \to \mu_0$ weakly as $\epsilon \downarrow 0$, so that B is the weak limit of a Brownian motion W conditioned on $\{|W_1| < \epsilon\}$. In this way, we say that "B is a Brownian motion conditioned on $B_1 = 0$", even though this is a 0-probability event.

Problem 12*. A Bessel process of dimension δ is given by the solution to the SDE:

$$dX_t = \frac{\delta - 1}{2} \cdot \frac{1}{X_t} dt + dB_t, \quad X_0 > 0$$

where B is a standard Brownian motion, at least up until the first time t that $X_t = 0$.

i. Show that $M_t = X_t^{2-\delta}$ is a continuous local martingale.

ii. For each a, let $\tau_a = \inf\{t \geq 0 : X_t = a\}$. For $a < X_0 < b$, compute $\mathbb{P}[\tau_a < \tau_b]$.

iii. Assume that $\delta < 2$. For $b > 1$, explain how one can condition on the event that $\tau_b < \tau_0$ using M.

iv. Using the previous part and the Girsanov theorem, describe the law of $X|_{[0,\tau_b]}$ conditioned on $\tau_b < \tau_0$.

v. Explain why, informally, the statement "A standard Brownian motion conditioned to be positive is a 3-dimensional Bessel process" is true.