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Problems marked with (†) may be handed in for marking (CCA pidgeonhole G/H). Problems marked with (⋆) are additional questions

Problem 1. Let X be a continuous semimartingale under $\mathbb{P}$, and let $\tilde{\mathbb{P}}$ be another probability measure on the same space such that $\tilde{\mathbb{P}} \ll \mathbb{P}$. Suppose that X is also a semimartingale under $\tilde{\mathbb{P}}$. Show that X has the same quadratic variation process under $\mathbb{P}$ and under $\tilde{\mathbb{P}}$.

Problem 2. Let b be bounded and measurable. Use Girsanov's theorem to construct a weak solution to the SDE

$$dX_t = b(X_t)dt + dB_t$$

over the finite (non-random) time interval $[0, T]$.

Problem 3. (†) Show that the SDE

$$dX_t = 3X_t^{1/3}dt + 3X_t^{2/3}dB_t, \quad X_0 = 0$$

has strong existence but not pathwise uniqueness.

Problem 4. Find the unique strong solution to the SDE

$$dX_t = \frac{1}{2}X_t dt + \sqrt{1 + X_t^2}dB_t, \quad X_0 = x.$$

(Hint: consider the change of variables $Y_t = \sinh^{-1}(X_t)$.)

Problem 5. (†) Construct a filtered probability space on which a Brownian motion B and an adapted process X are defined and such that

$$X_t = \int_0^t \frac{X_s}{s} ds + B_t, \quad B_0 = X_0 = 0.$$

Is X adapted to the filtration generated by B? Is B a Brownian motion in the filtration generated by X?

Problem 6. Let X be a solution of the SDE

$$dX_t = X_t g(X_t)dB_t$$

where g is bounded and $X_0 = x > 0$ is non-random.

i. By applying Ito's formula to

$$X_t \exp\left(-\int_0^t g(X_s)dB_s + \frac{1}{2}\int_0^t g^2(X_s)ds\right)$$

show that $\mathbb{P}(X_t > 0 \text{ for all } t \geq 0) = 1$.

ii. Show that $\mathbb{E}(X_t) = X_0$ for all $t \geq 0$.

iii. Fix a non-random time horizon $T > 0$. Show that there exists a measure $\widehat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ which is mutually absolutely continuous with respect to $\mathbb{P}$ and a $\widehat{\mathbb{P}}$-Brownian motion $\widehat{B}$ such that

$$dY_t = Y_t g(1/Y_t)d\widehat{B}_t$$

where $Y_t = 1/X_t$.

Problem 7. Consider the Cauchy problem for the quasi-linear parabolic equation

$$\frac{\partial V}{\partial t} = \frac{1}{2}\Delta V - \frac{1}{2}|\nabla V|^2 + k \quad \text{on} \quad (0,\infty)\times\mathbb{R}^d,$$

with $V(0,x) = 0$ for $x \in \mathbb{R}^d$ where $k\colon \mathbb{R}^d \to [0,\infty)$ is a continuous function. Suppose also that $V\colon [0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ is continuous on its domain, of class $C^{1,2}$ on $(0,\infty)\times\mathbb{R}^d$, and satisfies the quadratic growth condition for every $T > 0$:

$$-V(t,x) \le C + a|x|^2, \quad (t,x) \in [0,T]\times\mathbb{R}^d, \quad a < \frac{1}{2T}.$$

Show that $V(t,x)$ is given by

$$V(t,x) = -\log \mathbb{E}_x\left[\exp\left(-\int_0^t k(W_s)ds\right)\right]$$

for $t \ge 0$ and $x \in \mathbb{R}^d$.

Problem 8. Let $b\colon \mathbb{R}^d \to \mathbb{R}$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be bounded and continuous. For each n, j, set $t_j^n = n2^{-j}$ and $\psi_n(t) = t_j^n$ if $t \in [t_j^n, t_{j+1}^n)$. Assume that (X_0^n) is a tight sequence, and that X^n solves

$$X_t^n = X_0^n + \int_0^t b\left(X^n_{\psi_n(u)}\right) du + \int_0^t \sigma\left(X^n_{\psi_n(u)}\right) dB_u. \tag{1}$$

Show that for each $m, T > 0$ there exists a constant $C > 0$ such that

$$\mathbb{E}[|X_t^n - X_s^n|^{2m}] \le C(t-s)^m \quad \text{for all} \quad 0 \le s < t \le T. \tag{2}$$

Explain what it means for the sequence (X^n) to be tight in the space $C([0,T],\mathbb{R}^d)$. By looking at the proof of Kolmogorov's continuity criterion, explain why (2) implies that (X^n) is tight.

Problem 9. Consider the SDE

$$dX_t = X_t^2 dB_t.$$

i. By considering the process $\widetilde{X}_t = 1/|B_t - \xi|$ where B is a three-dimensional Brownian motion and ξ is a standard Gaussian in $\mathbb{R}^3$ independent of B, show that the SDE has a weak solution.

ii. Let $\Phi(s) = \int_{-\infty}^s e^{-t^2/2}\, dt/\sqrt{2\pi}$ be the Gaussian distribution function. Verify that both

$$u^1(t,x) = x\left(2\Phi(1/(x\sqrt{t})) - 1\right) \quad \text{and} \quad u^2(x,t) = x$$

solve the PDE

$$\frac{\partial u}{\partial t} = \frac{x^4}{2}\cdot\frac{\partial^2 u}{\partial x^2}, \quad u(0,x) = x \qquad \text{on } (0,\infty)\times(0,\infty).$$

iii. Which of these solutions corresponds to $u(t,x) = \mathbb{E}_x(X_t)$?

You may find it helpful to explain why SDEs with locally Lipschitz coefficients have uniqueness in law.

Problem 10*. The goal of this question is to show the following existence result for SDEs with *continuous* coefficients. Suppose $b, \sigma : \mathbb{R} \to \mathbb{R}$ are bounded and continuous, and $x \in \mathbb{R}$. Then, for any $T > 0$, there exists a weak solution $(X_t)_{t=0}^T$ to the SDE

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dB_t; \\ X_0 = x. \end{cases} \tag{3}$$

i. Define $\left(X_t^n\right)_{t=0}^T$ by (1), as in Question 8, with $X_0^n = x$, and using potentially different Brownian motions B^n. Let μ_n denote the law of $(X_n)_{t=0}^T$. Recalling Prohorov's Theorem, explain why there is a subsequence μ_{n_k} which converge weakly to a probability measure μ.

ii. By looking up *Skorohod's Representation Theorem*, explain why $(X_t^n)_{t=0}^T$ can be realised on a common filtered probability space, such that $X^n \to X$ uniformly, almost surely, where X has law μ.

iii. We claim that X is a weak solution to (3). Let $0 \le s \le t \le T$, and suppose $G : C[0,T] \to \mathbb{R}$ is bounded and continuous, such that $G(X)$ only depends on $X|_{[0,s]}$, and that $f \in C_b^2(\mathbb{R})$. Explain why it is sufficient to prove that

$$\mathbb{E}\left[\left(f(X_t) - f(X_s) - \int_s^t Lf(X_u)du\right) G(X)\right] = 0 \tag{4}$$

where

$$Lf(x) = b(x)f'(x) + \frac{1}{2}\sigma(x)^2 f''(x).$$

iv. Define, for $f \in C_b^2(\mathbb{R})$,

$$L_u^{n_k} f(x) = b\left(X_u^{n_k}\right) f'(x) + \frac{1}{2}\sigma\left(X_u^{n_k}\right)^2 f''(x).$$

Show that $L_u^{n_k} f\left(X_u^{n_k}\right) \to Lf(X_u)$ as $k \to \infty$, and deduce that

$$\int_s^t L_u^{n_k} f\left(X_u^{n_k}\right) du \to \int_s^t Lf(X_u)du.$$

v. Conclude.