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For a sequence $x = (x_n) \subset \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, recall the definitions of the p-norms,

$$\|x\|_p = \left(\sum_n |x_n|^p \right)^{1/p} \quad \text{for } p \in [1, \infty), \qquad \|x\|_\infty = \sup_n |x_n|,$$

and the sequence spaces

$$\ell^p = \{x = (x_n) \subset \mathbb{K} : \|x\|_p < \infty\}, \qquad \text{with } \|\cdot\|_p\text{-norm, for } p \in [1, \infty],$$
$$c_0 = \{x = (x_n) \subset \mathbb{K} : x_n \to 0 \text{ as } n \to \infty\}, \quad \text{with } \|\cdot\|_\infty\text{-norm.}$$

For $p \in [1, \infty]$, we use the convention that $1/0 = \infty$ and $1/\infty = 0$.

1. For $p, q, r \in (1, \infty)$ with $1/p + 1/q + 1/r = 1$ and $x \in \ell^p$, $y \in \ell^q$, $z \in \ell^r$, prove that

$$\|xyz\|_1 \le \|x\|_p \|y\|_q \|z\|_r.$$

2. For $p, q \in (1, \infty)$, $q > p$, show that the following inequalities hold on $\mathbb{K}^n$ and cannot be improved:

$$\|x\|_q \le \|x\|_p \le n^{\frac{1}{p} - \frac{1}{q}} \|x\|_q.$$

In particular, the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent on $\mathbb{K}^n$, but the constants depend on n.

3. Show that the space ℓ^p is complete for every $p \in [1, \infty]$.

4. For $p, q \in [1, \infty]$, show that $\ell^p \subset \ell^q$ if and only if $p \le q$.

5. For $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, show that $(\ell^p)^* = \ell^q$ in the sense that the spaces are isometrically isomorphic in a natural way.

6. Show that $c_0^* = \ell^1$ and that $(\ell^1)^* = \ell^\infty$ in the sense that the spaces are isometrically isomorphic.

7. Show that ℓ^p is separable if $p \in [1, \infty)$. Show that c_0 is separable. Show that ℓ^∞ is not separable.

8. Show that a normed vector space X is complete if and only if every absolutely convergent series is convergent. The latter means that $\sup_N \sum_{n=1}^N \|x_n\| < \infty$ implies that $\sum_{n=1}^N x_n$ converges as $N \to \infty$.

(Hint: to show that a Cauchy sequence (x_n) converges if every absolutely convergent series is convergent, first show that one may assume that $\|x_n - x_m\| \le 2^{-\min\{n,m\}}$.)

9. For a normed vector space X and bounded linear maps $T : X \to X$ and $S : X \to X$, show that TS is bounded and that $\|TS\| \le \|T\|\|S\|$. (Here TS is the composition of T and S.)

10. Let X be a normed vector space and define $\pi(x) = x/\|x\|$ for $x \in X \setminus \{0\}$. Either prove that then $\|\pi(x) - \pi(y)\| \le \|x - y\|$ whenever $\|x\|, \|y\| \ge 1$, or give an example in which this inequality is violated.

11. Let $x \in c_0$ and define $X = \{y \in c_0 : |y_n| \le |x_n|\}$. Show that X is compact in c_0.

12. Show that that space $C^1[0, 1]$ of continuously differentiable functions on $[0, 1]$ is complete in the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$ but incomplete in the norm $\|f\|_\infty$.

In applications, it is often useful to consider spaces with weights. Let $(\mu_n) \subset [0, \infty)$ be a nonnegative sequence of weights. Then define

$$\|x\|_{p,\mu} = \left(\sum_n |x_n|^p \mu_n \right)^{1/p} \quad \text{for } p \in [1, \infty),$$

and $\ell^p(\mu) = \{x = (x_n) \subset \mathbb{K} : \|x\|_{p,\mu} < \infty\}$.

13. For any sequence of weights μ, prove the Hölder inequality $\|xy\|_{1,\mu} \le \|x\|_{p,\mu} \|y\|_{q,\mu}$ if $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$.

14. If $\sum_n \mu_n < \infty$, show that $\ell^p(\mu) \supset \ell^q(\mu)$ if $p \le q$. Compare this with the case $\mu_n = 1$ in Exercise 5.