CIRM: Tricritical polymer density transition in MFT

<u>Ref</u>:
- (with G. Slade, forthcoming)
  comprehensive analysis of what I will outline

<u>Other</u>:
- (with D. Brydges, G. Slade)
  Introduction to a renormalisation group method
  Chapters 1 & 11
- (with T. Helmuth, A. Swan)
  The geometry of random walk isomorphisms
  Appendix A

<u>Plan</u>:
1. Model and features
2. Supersymmetric integral representation
3. Effective potential

1. Model and behaviour

Finite set Λ with edge weights $(\beta_{xy})_{x,y\in\Lambda}$.

Standard lattice model: $\beta_{xy} = 1_{x\sim y}$, $\Lambda \subset \mathbb{Z}^d$

Mean-field model: $\beta_{xy} = \frac{1}{N} 1_{x\neq y}$ ← from now on

SRW: (X_t) with generator $(\Delta f)_x = \sum_y \beta_{xy}(f_y - f_x)$.

Local time: $L_{T,x} = \int_0^T 1_{X_t = x}\, dt$

Expectation: $\mathbb{E}_a$ ← initial vertex

Interacting walk:

Weight: $p_\Lambda(L_T) = \prod_{x\in\Lambda} p(L_{T,x})$ for some $p: [0,\infty) \to [0,\infty)$.

$$\mathbb{E}_0^{(p)} F = \frac{1}{\chi^{(p)}} \int_0^\infty dT\ \mathbb{E}_0\left(F\ p_\Lambda(L_T)\right)$$

↑ walks have variable length T "grand canonical ensemble"

↑ normalisation = susceptibility

$$\chi^{(p)} = \int_0^\infty dT\ \mathbb{E}_0\left(p_\Lambda(L_T)\right)$$

Typical weights depend on parameters (chemical potential, interaction strength, etc.)

Susceptibility: $\chi^{(p)}$

End-to-end distance (on $\mathbb{Z}^d$):

$$\mathbb{E}|X|^2 = \frac{1}{\chi^{(p)}} \int_0^\infty dT \, \mathbb{E}^{(1)}\left(p_\Lambda(L_T)\, |X_T|^2\right)$$

Expected length:

$$\mathbb{E}^{(p)} L = \frac{1}{\chi^{(p)}} \int_0^\infty dT \; T \; \mathbb{E}^{(1)}\left(p_\Lambda(L_T)\right).$$

Density of walk:

$$\rho = \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \mathbb{E}^{(p)} L$$

$$e^{-g\sum_x L_{T,x}^2} = e^{-g\int_0^T\int_0^T 1_{X_s = X_t}} \qquad e^{-\nu \sum_x L_{T,x}} = e^{-\nu T}$$

<u>Examples:</u>

- weakly self-avoiding walk: $p(t) = e^{-gt^2 - \nu t}$, $g > 0$

  $\exists\, \nu_c < 0$ s.t. $\chi, \mathbb{E}L < \infty$ iff $\nu > \nu_c$

  <u>Conj.</u> As $\nu = \nu_c + t \downarrow \nu_c$: for $g > 0$ small:

$$\chi \sim \begin{cases} \frac{c}{t} & (d \geq 5) \\ c\frac{1}{t}(\log t)^{1/4} & (d=4) \\ c\, t^{-?} & (d=3) \\ c\, t^{-43/32} & (d=2) \end{cases}$$

$(d \geq 5)$ ← Brydges–Helmuth–Holmes

$(d=4)$ ← B–Brydges–Slade

$(d=3)$ ← open

$(d=2)$ ← open

and similar conjectures for other observables.

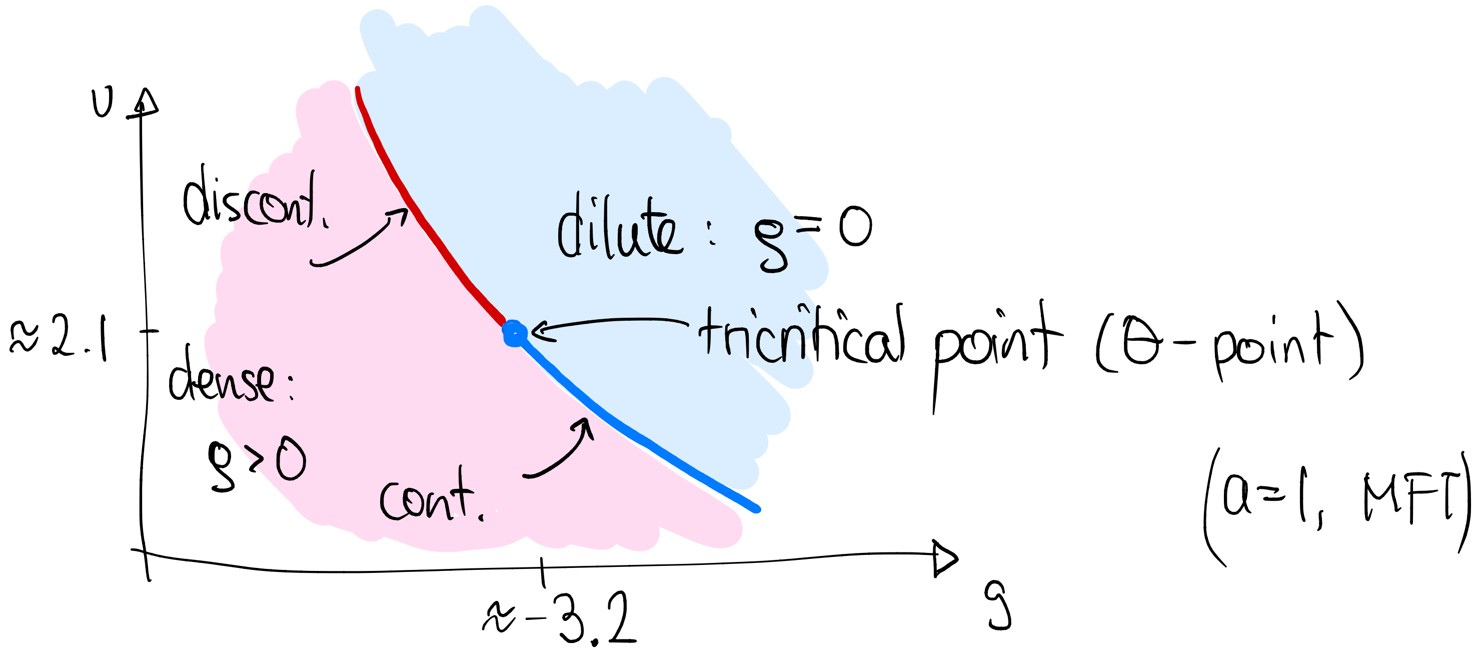
- Triple-well model: $p(t) = e^{-at^3 - gt^2 - \nu t}$, $a>0$ fixed

  triple intersections favoured ($a>0$), $a=1$ from now

  double intersections depreciated ($g<0$)

- Interacting self-avoiding walk ($\rightarrow$ Pétrélis)

Conjectured <u>phase diagram</u> for triple-well model ($d \geq 3$):



- For any point on blue curve the scaling limit is the same as that of the standard self-avoiding walk.

  The density is <u>continuous</u> across the blue curve.

- On the red curve (and the dense phase), the scaling limit is space filling.

  The density <u>jumps</u> across the red curve.

At the tricritical point the scaling limit is SRW in $d \geq 3$ and non-Gaussian (but different from SAW) in $d=2$.

<u>Thm</u> (w/ M. Lohmann, G. Slade). In $d=3$, $a>0$ small, there exists $g_c < 0$, $\nu_c > 0$ s.t.

$$G_{a,g_c,\nu_c}(x) \sim C|x|^{-1} \quad (\text{like SRW}).$$

$$\int_0^\infty \mathbb{E}_0\left(1_{X_T = x}\, P_{a,g_c,\nu_c}(L_T)\right) dT$$

Numerical predictions in $d=2$ (Duplantier, Saleur, ...)

Goal: derive this phase diagram in MFT.

$$G_{01} = \int_0^\infty dT \, \mathbb{E}_0 \left(1_{X_T = 1} \, p_\lambda(L_T) \right)$$

$$G_{00} = \int_0^\infty dT \, \mathbb{E}_0 \left(1_{X_T = 0} \, p_\lambda(L_T) \right)$$

Prop. Let

$$V(t) = t - \log\left(1 + \int_0^\infty p(s) e^{-s} \sqrt{\frac{t}{s}} \, I_1(2\sqrt{st}) \, ds \right).$$

(effective potential)

Then

$$G_{01} = \int_0^\infty e^{-NV(t)} \left(N V'(t)(1 - V'(t)) + 2V''(t) \right)(1 - V'(t)) \, t \, dt$$

$$G_{00} = \int_0^\infty e^{-NV(t)} \left(\cdots \right.$$

$$\mathbb{E}L = \int_0^\infty e^{-NV(t)} \left(\cdots \right.$$

looks simpler in supersymmetric form

Upshot: Given interaction p (say $p(t) = e^{-t^3 - gt^2 - \nu t}$)

compute V (one-dim. integral!)

Laplace method then gives asymptotic behaviour.

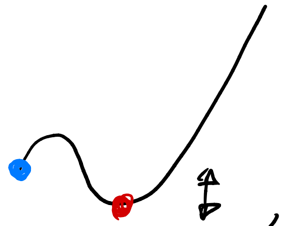
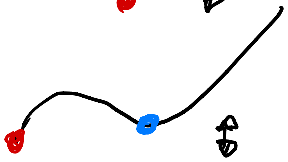
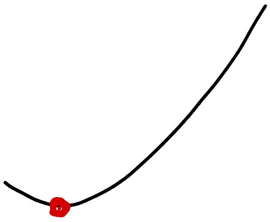
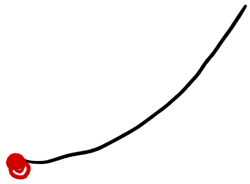
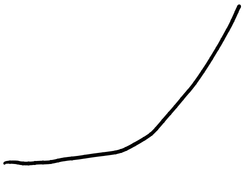
Basic version of Laplace method:

$$\frac{\int e^{-NV(t)} F(t) \, dt}{\int e^{-NV(t)} \, dt} \sim F(t_0)$$

unique minimum of V

(appropriate assumptions)

How does V look?

g	v		
-4.4	4.22		dense phase near first-order curve
-4.4	4.25		dilute phase near first-order curve
-2.7	1.2		dense phase near second-order curve
-2.7	1.4		dilute phase near second-order curve
tricritical point			

Defn.

- dilute phase: $V'(0)>0$, unique global min. at 0
- second-order curve: $V'(0)=0$, $V''(0)>0$, unique gl. min. at 0
- tricritical point: $V'(0)=0=V''(0)$, $V'''(0)>0$, uniq. gl. min
- dense phase: unique global min. $V(t_0)<0$ at $t_0>0$, $V''(t_0)>0$
- first order curve: $V(t)\geq 0\ \forall t$, $V(t_0)=0$ at $t_0>0$, $V''(t_0)>0$

Thm.

$$\mathbb{E}L \sim \begin{cases} \frac{1}{1-V'(0)} (\cdots) & \text{dilute phase} \\ N^{1/2} (\cdots) & \text{second order curve} \\ N^{2/3} (\cdots) & \text{tricrit. pt.} \\ N\, V(t_0) & \text{dense phase and first order curve} \end{cases}$$

and similar formulas hold for susceptibility and two-point function.

Follows from detailed analysis of Laplace integrals (→ forthcoming preprint w/ G. Slade)

Next: derivation of Laplace integral representation.

Idea: simple block spin transformation

This concept (in less explicit form) is also the starting point for much more involved analysis on $\mathbb{Z}^d$ when we can do it.

2. Derivation of effective potential.

Exterior (Grassmann) algebra Ω^{2N} :

bar is only notation

symbols $\psi_1, \bar{\psi}_1, \dots, \psi_N, \bar{\psi}_N$ all anticommuting

$$\psi_1\psi_2 = -\psi_2\psi_1, \quad \psi_1\bar{\psi}_1 = -\bar{\psi}_1\psi_1, \quad \psi_1\psi_1 = 0, \quad \dots$$

Fermionic derivative

$$\partial_{\psi_1}(\psi_1\bar{\psi}_4 \cdots) = \bar{\psi}_4 \cdots$$

need to commute ψ_1 to the left first

Fact. Let A be a symm. $N\times N$ matrix. Then

$$\det A = \partial_{\psi_N}\partial_{\bar{\psi}_N}\cdots\partial_{\psi_1}\partial_{\bar{\psi}_1}\underbrace{e^{-(\psi, A\bar{\psi})}}_{\sum_{n=0}^{N}\frac{(-1)^N}{N!}\left(\sum_{i,j}\psi_i A_{ij}\bar{\psi}_j\right)^N}$$

$$= \text{———}\ \text{''}\ \text{———}\ \frac{(+1)^N}{N!}\left(\sum_{i,j}\bar{\psi}_i A_{ij}\psi_j\right)^N \quad \checkmark$$

Fact. Assume A has pos.-def. hermitian part. Then

$$\frac{1}{\det A} = \int_{\mathbb{C}^N}\prod_{x=1}^{N}\underbrace{\frac{d\phi_x d\bar{\phi}_x}{2\pi i}}_{\frac{du_x dv_x}{\pi}} e^{-(\phi, A\bar{\phi})}$$

$$\phi_x = u_x + iv_x \quad \bar{\phi}_x = u_x - iv_x$$

Cor. If A has pos.-def. hermitian part, then

$$1 = \underbrace{\int_{\mathbb{C}^N} \prod_{x=1}^{N} \frac{d\phi_x d\bar{\phi}_x \partial_{\psi_x} \partial_{\bar{\psi}_x}}{2\pi i}}_{\text{superintegral } \int} \underbrace{e^{-(\phi, A\bar{\phi}) - (\psi, A\bar{\psi})}}_{\text{integrand is a form}} .$$

can be identified with diff. forms

Forms are (noncomm.) polynomials in the $\psi_x, \bar{\psi}_x$ whose coefficients are C^∞ functions of the $\phi_x, \bar{\phi}_x$.

Example: $\underset{\substack{\uparrow \\ \text{function on } \mathbb{R}^{2N} \\ \text{degree } 0}}{\bar{\phi}_i \phi_i} + \underset{\substack{\uparrow \\ \text{degree } 2}}{\bar{\psi}_i \psi_i} = \tau_i$ is an even form.

For $f : \mathbb{R}^p \to \mathbb{R}$ smooth, and $\omega_1, \dots, \omega_p$ even forms, set

$$f(\omega_1, \dots, \omega_p) = f(\omega_1|_0, \dots, \omega_p|_0) + \partial_1 f(\omega_1|_0, \dots, \omega_p|_0)(\omega_1 - \omega_1|_0)$$
$$+ \dots \qquad + \dots$$

formal Taylor expansion
finite since ψ's are nilpotent!

Example: $e^{-\tau_1} = e^{-\bar{\phi}_1\phi_1 - \bar{\psi}_1\psi_1} = e^{-\bar{\phi}_1\phi_1}(1 - \bar{\psi}_1\psi_1)$.

$$e^{-\tau_1 - \tau_2} = e^{-\bar{\phi}_1\phi_1 - \bar{\phi}_2\phi_2}\left(1 - \bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2 + \bar{\psi}_1\psi_1\bar{\psi}_2\psi_2\right).$$

Example:

$$\int_{\mathbb{C}} \frac{d\phi\, d\bar\phi\, \partial_\psi \partial_{\bar\psi}}{2\pi i} f(\tau) = \int_{\mathbb{C}} \frac{d\phi\, d\bar\phi\, \partial_\psi \partial_{\bar\psi}}{2\pi i} \Big(\underbrace{f(\phi\bar\phi)}_{0} + f'(\phi\bar\phi) \underbrace{\psi\bar\psi}_{-\bar\psi\psi} \Big)$$

$$= -\int_{\mathbb{C}} \frac{d\phi\, d\bar\phi}{2\pi i} f'(\phi\bar\phi)$$

$$= -\int_{\mathbb{R}^2} \frac{du\, dv}{\pi} f'(u^2+v^2)$$

$$= -\int_0^\infty dt\, f'(t) = f(0).$$

This is an example of the following 'localisation theorem'.

The forms τ_i are formally invariant under exchanging the ψ and ϕ (supersymmetry):

$$Q\tau_i = 0$$

since we first need to commute ψ to the left.

where $$Q = \sum_{i=1}^N \left(\psi \frac{\partial}{\partial \phi} + \bar\psi \frac{\partial}{\partial \bar\phi} - \phi \frac{\partial}{\partial \psi} + \bar\phi \frac{\partial}{\partial \bar\psi} \right)$$

<u>Thm</u>. For $f \in \Omega^{2N}(\mathbb{R}^{2N})$ smooth with sufficient decay and $Qf = 0$,

$$\int f = f(0)\big|_0$$

formally set $\psi = 0$ (take degree-0 part)

evaluate $\phi = 0$

superintegral

related: Duistermaat-Heckman thm

Example:

$$\int e^{-\sum_{ij}(\bar\Phi_i\phi_j + \bar\psi_i\psi_j)A_{ij}} = 1$$

Thm

$$\int_0^\infty \mathbb{E}_a\left(1_{X_T=b}\,\rho(L_T)\right) = \int e^{-(\phi,-\Delta\bar\phi)-(\psi,-\Delta\bar\psi)} \rho(\phi\bar\phi+\psi\bar\psi)\,\bar\phi_a\phi_b$$

Proof. Let

Laplacian acting on x variable

$$\mathcal{L}g(x,\ell) = \Delta g(x,\ell) + \frac{d}{d\ell_x} g(x,\ell)$$

generator of process (X_t, L_t).

$$\Rightarrow \int e^{(\phi,\Delta\bar\phi)+(\psi,\Delta\bar\psi)} \sum_x \bar\phi_a\phi_x \underbrace{\mathcal{L}g(x,\phi\bar\phi+\psi\bar\psi)}_{\bar\phi_x\Delta g(x,\phi\bar\phi+\psi\bar\psi)+\frac{\partial}{\partial\phi_x}g(x,\phi\bar\phi+\psi\bar\psi)}$$

$$\overset{\text{IBP}}{=} \int e^{(\phi,\Delta\bar\phi)+(\psi,\Delta\bar\psi)} g(a,\phi\bar\phi+\psi\bar\psi)$$

$$\overset{\text{SUSY}}{=} g(a,0).$$

$Q(\cdots)=0$

Invert $\mathcal{L}$: $g(x,\ell) = \mathbb{E}_{x,\ell}\left(1_{X_T=b}\,\rho(L_T)\right)$

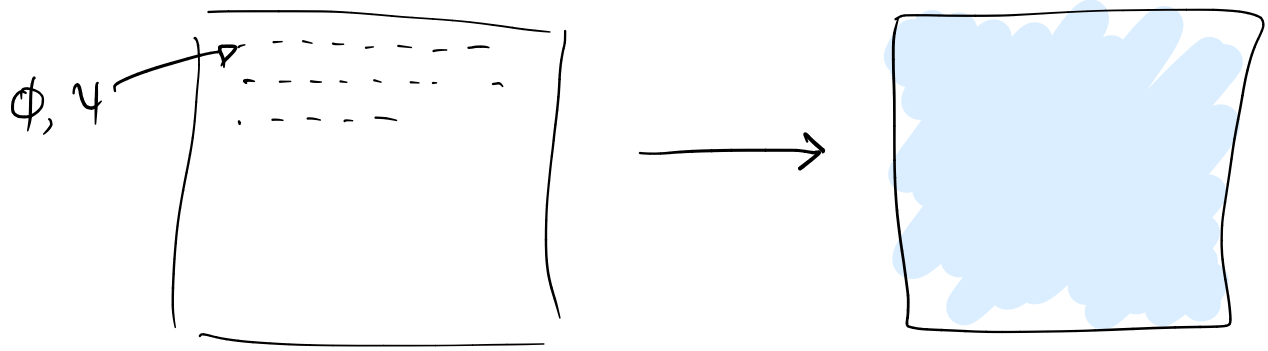
$$\Rightarrow \quad \mathcal{L}g(x,\ell) = \frac{d}{dT} g(x,\ell)$$

$$\Rightarrow \int e^{(\phi,\Delta\bar\phi)+(\psi,\Delta\bar\psi)} \bar\phi_a\phi_b\, \rho(\phi\bar\phi+\psi\bar\psi) = \int_0^\infty \mathbb{E}_a\left(1_{X_t=b}\,\rho(L_T)\right).$$

Lemma. If $(\Delta f)_x = \frac{1}{N}\sum_y (f_x - f_y)$ then

$$e^{-(\phi,-\Delta\bar\phi)-(\psi,-\Delta\bar\psi)} = \int \frac{d\zeta\, d\bar\zeta\, \partial_\xi \partial_{\bar\xi}}{2\pi i}\left[e^{-(\phi-\zeta,\bar\phi-\bar\zeta)}\, e^{-(\psi-\xi,\bar\psi-\bar\xi)}\right]$$

$\mathbb{R}^{2N}$ $\quad$ Ω^{2N} $\quad$ $\mathbb{R}^2$ $\quad$ Ω^2



microscopic variables ϕ_x, ψ_x $\qquad$ block variable ζ, ξ

Proof. Note $-\Delta$ is the orthogonal proj. onto $\{f : \sum_x f = 0\}$.

Let $Af = \frac{1}{N}\sum_x f_x$.

$$\Rightarrow (\phi-\zeta, \bar\phi-\bar\zeta) = (\underbrace{\phi - A\phi}_{\text{mean } 0} - \underbrace{(\zeta - A\phi)}_{\text{constant}},\ \bar\phi - A\bar\phi - (\bar\zeta - A\bar\phi))$$

$$= \underbrace{(\phi - A\phi, \bar\phi - A\bar\phi)}_{(\phi, -\Delta\bar\phi)} + \underbrace{(\zeta - A\phi, \bar\zeta - A\bar\phi)}_{N|\zeta - A\phi|^2}$$

$$\Rightarrow \text{RHS} = e^{-(\phi,-\Delta\bar\phi)-(\psi,-\Delta\bar\psi)} \times \underbrace{\int \frac{d\zeta\, d\bar\zeta\, \partial_\xi \partial_{\bar\xi}}{2\pi i}\, e^{-N\left(|\zeta - \cancel{A\phi}|^2 + (\xi - \cancel{A\psi})(\bar\xi - \cancel{A\bar\psi})\right)}}_{=1}$$

Substituting this into the previous theorem:

$$\int_0^\infty \mathbb{E}_a\left(1_{X_T=b}\, \rho(L_T)\right)$$

$$= \int_{\mathbb{R}^{2N}} \prod_x \frac{d\phi_x d\bar{\phi}_x \partial_{\psi_x} \partial_{\bar{\psi}_x}}{2\pi i} e^{-(\phi,-\Delta\bar{\phi})-(\psi,-\Delta\bar{\psi})} \rho(\phi\bar{\phi}+\psi\bar{\psi})\, \bar{\phi}_a \phi_b$$

$$= \int_{\mathbb{R}^2} \frac{d\zeta d\bar{\zeta}\partial_\xi \partial_{\bar{\xi}}}{2\pi i} \underbrace{\left[\int_{\mathbb{R}^{2N}} \prod_x \frac{d\phi_x d\bar{\phi}_x \partial_{\psi_x}\partial_{\bar{\psi}_x}}{2\pi i} e^{-\sum_x\left((\phi_x-\zeta)(\bar{\phi}_x-\bar{\zeta})-(\psi_x-\xi)(\bar{\psi}_x-\bar{\xi})\right)} \prod_x \rho(\phi_x\bar{\phi}_x+\psi_x\bar{\psi}_x)\, \bar{\phi}_a\phi_b \right]}_{\text{factorises into product!}}$$

Defn. Define $V: \mathbb{R} \to \mathbb{R}$ by

$$e^{-V(\zeta\bar{\zeta}+\xi\bar{\xi})} \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \frac{d\phi d\bar{\phi}\partial_\psi\partial_{\bar{\psi}}}{2\pi i} e^{-(\phi-\zeta)(\bar{\phi}-\bar{\zeta})-(\psi-\xi)(\bar{\psi}-\bar{\xi})} \rho(\phi\bar{\phi}+\psi\bar{\psi}) \qquad (*)$$

Fact. RHS is indeed function of $\zeta\bar{\zeta}+\xi\bar{\xi}$.

<u>Prop</u>. $V(t) = t - \log\left(p(0) + \int_0^\infty p(s) e^{-s} \sqrt{\frac{t}{s}}\, I_1(2\sqrt{ts})\, ds\right)$.

Bessel function

<u>Proof</u>. By the fact, suffices to set $\xi = \bar{\xi} = 0$.

$\Rightarrow e^{-(\phi-\xi)(\bar{\phi}-\bar{\xi}) - (\psi-\xi)(\bar{\psi}-\bar{\xi})} \, p(\phi\bar{\phi} + \psi\bar{\psi})$

$= e^{-|\xi|^2 + \xi\bar{\phi} + \bar{\xi}\phi} \, \tilde{p}(\phi\bar{\phi} + \psi\bar{\psi})$ where $\tilde{p}(t) = e^{-t} p(t)$

$= \text{———}\,''\,\text{———} \left(\tilde{p}(\phi\bar{\phi}) - \tilde{p}'(\phi\bar{\phi})\psi\bar{\psi}\right)$

$\Rightarrow$ RHS of $(*) = -e^{-|\xi|^2} \int_{\mathbb{C}} \frac{d\phi\, d\bar{\phi}}{2\pi i} e^{\xi\bar{\phi} + \bar{\xi}\phi} \tilde{p}'(\phi\bar{\phi})$

$\rightarrow$ convert to polar coordinates and use defn. of Bessel functions

<u>Exercise</u>. • Find expressions in terms of V if $\bar{\Phi}_a$ or Φ_b or $\bar{\Phi}_a\Phi_a$ are inserted inside the integral in $(*)$.

• Using the expressions derive integral representation for two point function.