A SIMPLE PROBABILISTIC PROOF OF STOLARSY'S INVARIANCE PRINCIPLE

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This note accompanies the main article "Permutation p-value approximation via generalized Stolarsky invariance" by Hera Y. He, Kinjal Basu, Qingyuan Zhao, and Art B. Owen, available at arXiv 1603.02757. Here we give a stand-alone proof of Stolarsky's invariance principle.

Let σ be the uniform probability measure on the d-dimensional sphere $\mathbb{S}^d$ and U be the (discrete) uniform distribution supported on $x_1, \dots, x_N \in \mathbb{S}^d$. For a probability measure P, we use the notation $P_Y[f(Y)] = \mathrm{E}_{Y\sim P}[f(Y)]$ if Y is a random variable. We use a hollow circle $\circ$ to indicate multiple expectation/integral. For example $P_X \circ P_Y[f(X,Y)] = P_X\big[P_Y[f(X,Y)]\big]$.

Lemma 1. *L_2 discrepancy formula with fixed cap size:*

$$\begin{aligned}(1)\qquad &\sigma_Y\left[\big|U_X[I(X'Y \geq t)] - \sigma_X[I(X'Y \geq t)]\big|^2\right]\\ =&U_{X_1} \circ U_{X_2} \circ \sigma_Y[I(X_1'Y \geq t, X_2'Y \geq t)] - \sigma_{X_1} \circ \sigma_{X_2} \circ \sigma_Y[I(X_1'Y \geq t, X_2'Y \geq t)].\end{aligned}$$

Proof. The key is to think $Z(Y) = U_X[I(X'Y \geq t)]$ as a random variable and notice that $\sigma_X[I(X'Y \geq t)]$ is a constant (only depends on t). Then the expectation of $Z(Y)$ is

$$\sigma_Y[Z(Y)] = U_X[\sigma_Y[I(X'Y \geq t)]] = \sigma_Y[I(X'Y \geq t)] = \sigma_X[I(X'Y \geq t)].$$

The first equality uses Fubini's theorem. Therefore the left hand side of (1) is the variance of $Z(Y)$, and we complete the proof by showing the two terms on the right hand side are, respectively, equal to $\mathrm{E}[Z(Y)^2]$ and $\big\{\mathrm{E}[Z(Y)]\big\}^2$:

$$\begin{aligned}\mathrm{E}[Z(Y)^2] &= \sigma_Y\left[\big|U_X[I(X'Y \geq t)]\big|^2\right]\\ &= \sigma_Y\big[U_{X_1}[I(X_1'Y \geq t)] \cdot U_{X_2}[I(X_2'Y \geq t)]\big]\\ &= \sigma_Y \circ U_{X_1} \circ U_{X_2}[I(X_1'Y \geq t) \cdot I(X_2'Y \geq t)]\\ &= U_{X_1} \circ U_{X_2} \circ \sigma_Y[I(X_1'Y \geq t, X_2'Y \geq t)].\\ \big\{\mathrm{E}[Z(Y)]\big\}^2 &= \Big\{\sigma_Y\big[U_X[I(X'Y \geq t)]\big]\Big\}^2\\ &= \Big\{U_X\big[\sigma_Y[I(X'Y \geq t)]\big]\Big\}^2\\ &= \Big\{\sigma_X\big[\sigma_Y[I(X'Y \geq t)]\big]\Big\}^2\\ &= \sigma_{X_1}\big[\sigma_Y[I(X_1'Y \geq t)]\big] \cdot \sigma_{X_2}\big[\sigma_Y[I(X_2'Y \geq t)]\big]\\ &= \sigma_{X_1} \circ \sigma_{X_2} \circ \sigma_Y[I(X_1'Y \geq t, X_2'Y \geq t)].\end{aligned}$$

Again, we have repeatedly used Fubini's theorem. □

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Lemma 2. *Given* $x, z \in \mathbb{S}^d$,

$$\sigma_Y\left[\min(Y'x, Y'z)\right] = \int_{\mathbb{S}^d} \min(y'x, y'z)\,\mathrm{d}\sigma_d(y) = -\frac{\|x-z\|}{dB(\frac{1}{2},\frac{d}{2})}. \tag{2}$$

Proof. Without loss of generality, we can rotate x and z so that

$$x = (s, \sqrt{1-s^2}, 0, \dots, 0),\ \ z = (-s, \sqrt{1-s^2}, 0, \dots, 0).$$

Any point $y \in \mathbb{S}^d$ can be represented by the polar coordinates:

$$y = (\cos\theta_1, \sin\theta_1\cos\theta_2, \cdots),\ \ 0 \le \theta_1, \theta_2 \le \pi.$$

Hence $y'x \ge y'z$ is equivalent to $\theta_1 \le \pi/2$. We have

$$\begin{aligned}
&\int_{\mathbb{S}^d} \min(y'x, y'z)\,\mathrm{d}\sigma_d(y)\\
=&2\int_0^{\pi/2}\int_0^{\pi}(-s\cos\theta_1 + \sqrt{1-s^2}\sin\theta_1\cos\theta_2)\frac{(\sin\theta_1)^{d-1}(\sin\theta_2)^{d-2}}{B(\frac{1}{2},\frac{d}{2})B(\frac{1}{2},\frac{d-1}{2})}d\theta_2 d\theta_1.
\end{aligned}$$

The second term integrates to 0 because $\cos\theta_2(\sin\theta_2)^{d-2}$ is odd with respect to $\pi/2$. Hence

$$\begin{aligned}
&\int_{\mathbb{S}^d} \min(y'x, y'z)\,\mathrm{d}\sigma_d(y)\\
=&-2s\int_0^{\pi/2}\int_0^{\pi}\cos\theta_1\frac{(\sin\theta_1)^{d-1}(\sin\theta_2)^{d-2}}{B(\frac{1}{2},\frac{d}{2})B(\frac{1}{2},\frac{d-1}{2})}\,\mathrm{d}\theta_2\,\mathrm{d}\theta_1\\
=&-\frac{2s}{B(\frac{1}{2},\frac{d}{2})B(\frac{1}{2},\frac{d-1}{2})}\left(\int_0^{\pi/2}\cos\theta_1(\sin\theta_1)^{d-1}\,\mathrm{d}\theta_1\right)\left(\int_0^{\pi}(\sin\theta_2)^{d-2}\,\mathrm{d}\theta_2\right)\\
=&-\frac{2s}{B(\frac{1}{2},\frac{d}{2})B(\frac{1}{2},\frac{d-1}{2})}\cdot\frac{1}{d}\cdot B(\frac{1}{2},\frac{d-1}{2})\\
=&-\frac{2s}{dB(\frac{1}{2},\frac{d}{2})} = -\frac{\|x-z\|}{dB(\frac{1}{2},\frac{d}{2})}.
\end{aligned}$$

□

Now we are set to prove Stolarsky's invariance principle. Let L be the Lebesgue measure on $[-1, 1]$.

$$\begin{aligned}
&L_T\left[\sigma_Y\left[\left|U_X[I(X'Y \ge t)] - \sigma_X[I(X'Y \ge t)]\right|^2\right]\right]\\
=&L_T\left[U_{X_1}\circ U_{X_2}\circ\sigma_Y\left[I(X_1'Y \ge T, X_2'Y \ge T)\right] - \sigma_{X_1}\circ\sigma_{X_2}\circ\sigma_Y\left[I(X_1'Y \ge T, X_2'Y \ge T)\right]\right]\\
=&U_{X_1}\circ U_{X_2}\circ\sigma_Y\left[L_T[I(X_1'Y \ge T, X_2'Y \ge T)]\right] - \sigma_{X_1}\circ\sigma_{X_2}\circ\sigma_Y\left[L_T[I(X_1'Y \ge T, X_2'Y \ge T)]\right]\\
=&U_{X_1}\circ U_{X_2}\circ\sigma_Y\left[\min(Y'X_1, Y'X_2)\right] - \sigma_{X_1}\circ\sigma_{X_2}\circ\sigma_Y\left[\min(Y'X_1, Y'X_2)\right]\\
=&\frac{1}{dB(\frac{1}{2},\frac{d}{2})}\left\{\sigma_{X_1}\circ\sigma_{X_2}\left[\|X_1 - X_2\|\right] - U_{X_1}\circ U_{X_2}\left[\|X_1 - X_2\|\right]\right\}.
\end{aligned}$$

In the equalities above, we have used, respectively, Lemma 1, Fubini's theorem, a simple integration of indicator variable, and Lemma 2.