Exercise 1

a) Let $B$ be a random variable taking values in $[a, b]$. Show that $B - E[B] \in sG\left(\frac{(b-a)^2}{4}\right)$.

b) Let $X_1, \ldots, X_n$ be independent random variables such that $X_i \in [a_i, b_i]$, and $\bar{X} = \frac{1}{n} \sum X_i$. Provide an upper bound on $P(\bar{X} - E[\bar{X}] > t)$.

Exercise 2

Let $X_1$ and $X_2$ be two random variables such that $X_1 \in sG(\sigma_1^2)$ and $X_2 \in sG(\sigma_2^2)$ (note that no independence is assumed). Show that $X_1 + X_2 \in sG(2(\sigma_1^2 + \sigma_2^2))$.

Exercise 3

We recall that for $X \in sG(\sigma^2)$, there is an explicit bound on $E[|X|^k]$. Show that conversely, for any random variable $Z$ with $E[Z] = 0$, if $E[|Z|^k] \leq 2(\sqrt{2}\sigma)^k \Gamma\left(\frac{k}{2} + 1\right)$ for some $\sigma > 0$ and all $k \geq 2$, then $Z \in sG(c\sigma^2)$ for some constant $c \geq 1$.

Exercise 4

a) Show that there exists a random variable $X \in sG(1)$ such that $X \notin sG(\tau^2)$ for all $\tau \in (0, 1)$.

b) Show that for all $\varepsilon \in (0, 1)$, there exists $X$ with the above property such that $E[X^2] = \varepsilon$.

c) Show that it is not possible to obtain a bound valid for all $X \in sG(1)$ and all $t > 0$, of the form

$$P(X > t) \leq e^{-\frac{t^2}{2E[X^2]}}.$$

Exercise 5

Let $X \in sG(\sigma^2)$, and $\Delta_4 = E[X^4]^{1/2}$.

a) Show that $E[X^2] \leq \Delta_4 \leq 4\sigma^2$. Give an example where $\Delta_4$ is much smaller than $\sigma^2$.

b) Show that for all $t > 0$, it holds that

$$P(X > t) \leq \exp\left( -\frac{t^2}{2\Delta_4 + 2\sqrt{2}\sigma t} \right).$$
Exercise 6

Let $X \in sG_d(\sigma^2)$. Show that for all $\delta \in (0, 1)$, it holds with probability $1 - \delta$ that

$$
\|X\|_1 \leq C\sigma d + \sigma \sqrt{2d \log(1/\delta)},
$$

for some constant $C > 0$.

Exercise 7

Let $A$ be a random matrix of $R^{m \times n}$ with independent coefficients such that $A_{ij} \in sG(\sigma^2)$.

a) Show that $A \in sG_{m \times n}(\sigma^2)$, i.e. that $\text{Tr}(AU) \in sG(\sigma^2)$ for all $\|U\|^2_F = \sum_{i,j} U_{ij}^2 = 1$.

b) Show that for some constant $C > 0$

$$
\mathbb{E}[\|A\|] \leq C\sigma(\sqrt{m} + \sqrt{n}),
$$

where $\|A\|$ is the operator norm of $A$ defined by $\max_{x \in R^m} \|Ax\|_2/\|x\|_2$.

Exercise 8

Consider the following testing problem, for $Z \in sG_d(1)$ and some $S \subset R^d$

$$
H_0 : X = Z \\
H_1 : X = \theta + Z, \theta \in S.
$$

a) For $m \geq 2$, let $d = \left(\binom{m}{2}\right)$ and $X \in R^d$ be the weights observed along the edges of a complete graph with $m$ vertices. For some $\mu > 0$, let $S$ be defined as

$$
S = \{\mu 1_{E_s} | \text{ for all } 1 \leq s \leq m\},
$$

where $E_s$ is the set of edges of a star graph, i.e. a graph whose only edges are those between vertex $s$ and the $m - 1$ other vertices. For all $\delta \in (0, 1)$, describe a test that has probability of error less than $\delta$ when

$$
\mu \geq 2\sqrt{\frac{2 \log(m)}{m}} + 2\sqrt{\frac{2 \log(1/\delta)}{m}}.
$$

Is the test easily implementable?

b) With the same setting as above, consider for $k \leq m$

$$
S = \{\mu 1_{K_S} | \text{ for all } S, |S| = k\},
$$

where $K_S$ is the set of edges of a clique, i.e. a graph whose only edges are those between vertices in $S$. For all $\delta \in (0, 1)$, describe a test that has probability of error less than $\delta$ when

$$
\mu \geq C'\sqrt{\frac{2 \log(me/k)}{k-1}} + C'\sqrt{\frac{2 \log(1/\delta)}{k(k-1)}},
$$

for some constants $C, C' > 0$. Is the test easily implementable?