**MATHEMATICAL TRIPOS: PART IA** **Lent 2021**

**PROBABILITY** **Perla Sousi**

**Example Sheet 2 (of 4)**

**1.** A coin with probability $p \in [0, 1]$ of heads is tossed $n$ times. Let $E$ be the event 'a head is obtained on the first toss' and $F_k$ the event 'exactly $k$ heads are obtained'. For which pairs of non-negative integers $(n, k)$ are $E$ and $F_k$ independent?

**2.** The events $A$ and $B$ are independent. Show that the events $A^c$ and $B$ are independent, and that the events $A^c$ and $B^c$ are independent.

**3.** Independent trials are performed, each with probability $p$ of success. Let $P_n$ be the probability that $n$ trials result in an even number of successes. Show that

$$P_n = \tfrac{1}{2}(1 + (1 - 2p)^n).$$

**4.** Two darts players $A$ and $B$ throw alternately at a board and the first to score a bull wins the contest. The outcomes of different throws are independent and on each of their throws $A$ has probability $p_A$ and $B$ has probability $p_B$ of scoring a bull. If $A$ has first throw, calculate the probability $p$ that $A$ wins the contest.

**5.** Consider the probability space $\Omega = \{0, 1\}^3$ with equally likely outcomes.

(a) Show that there are 70 different Bernoulli random variables of parameter $1/2$ that can be defined on $\Omega$.

(b) How many Bernoulli random variables of parameter $1/3$ can be defined on $\Omega$?

(c) What is the length of the longest sequence of independent Bernoulli random variables of parameter $1/2$ that can be defined on $\Omega$?

**6.** Suppose that $X$ and $Y$ are independent Poisson random variables with parameters $\lambda$ and $\mu$ respectively. Find the distribution of $X + Y$. Prove that the conditional distribution of $X$, given that $X + Y = n$, is binomial with parameters $n$ and $\lambda/(\lambda + \mu)$.

**7.** The number of misprints on a page has a Poisson distribution with parameter $\lambda$, and the numbers on different pages are independent.

(a) What is the probability that the second misprint will occur on page $r$?

(b) A proof-reader studies a single page looking for misprints. She catches each misprint (independently of others) with probability $p \in [0, 1]$. Let $X$ be the number of misprints she catches and let $Y$ be the number she misses. Find the distributions of the random variables $X$ and $Y$ and show they are independent.

**8.** Let $X_1, \dots, X_n$ be independent identically distributed random variables with mean $\mu$ and variance $\sigma^2$. Find the means of the random variables

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2.$$

**9.** In a sequence of $n$ independent trials the probability of a success at the $i$th trial is $p_i$. Let $N$ denote the total number of successes. Find the mean and variance of $N$.

**10.** Liam's bowl of spaghetti contains $n$ strands. He selects two ends at random and joins them together. He repeats this until no ends are left. What is the expected number of spaghetti hoops in the bowl?

**11.** Sarah collects figures from cornflakes packets. Each packet contains one of $n$ distinct figures. Each type of figure is equally likely. Show that the expected number of packets Sarah needs to buy to collect a complete set of $n$ is

$$n\sum_{i=1}^{n}\frac{1}{i}.$$

**12.** Let $a_1, a_2, \dots, a_n$ be a ranking of the yearly rainfalls in Cambridge over the next $n$ years. Assume that $a_1, a_2, \dots, a_n$ is a random permutation of $1, 2, \dots, n$. Say that $k$ is a record year if $a_k < a_i$ for all $i < k$. Thus the first year is always a record year. Let $Y_i = 1$ if $i$ is a record year and 0 otherwise. Find the distribution of $Y_i$ and show that $Y_1, Y_2, \dots, Y_n$ are independent. Calculate the mean and variance of the number $N$ of record years in the next $n$ years.

**13.** Let $s \in (1, \infty)$ and let $X$ be a random variable in $\{1, 2, \dots\}$ with distribution

$$\mathbb{P}(X = n) = n^{-s}/\zeta(s)$$

where $\zeta(s)$ is a suitable normalizing constant. For each prime number $p$ let $A_p$ be the event that $X$ is divisible by $p$. Find $\mathbb{P}(A_p)$ and show that the events $(A_p : p \text{ prime})$ are independent. Deduce that

$$\prod_p \left(1 - \frac{1}{p^s}\right) = \frac{1}{\zeta(s)}.$$