Example sheet 4

1 Brownian motion

Exercise 1.1. (i) Let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{R}^2$ starting from (x, y). Compute the distribution of B_T, where

$$T = \inf\{t \geq 0 : B_t \notin H\}$$

and where H is the upper half plane $\{(x, y) : y > 0\}$.

(ii) Show that, for any bounded continuous function $u : \overline{H} \to \mathbb{R}$, harmonic in H, with $u(x, 0) = f(x)$ for all $x \in \mathbb{R}$, we have

$$u(x, y) = \int_{\mathbb{R}} f(s) \frac{1}{\pi} \frac{y}{(x - s)^2 + y^2} \, ds.$$

Exercise 1.2 (Brownian bridge)**.** Let $(B_t, 0 \leq t \leq 1)$ be a standard Brownian motion in 1 dimension. We let $(Z_t^y = yt + (B_t - tB_1), 0 \leq t \leq 1)$ for any $y \in \mathbb{R}$ and call it the Brownian bridge from 0 to y. Let W_0^y be the law of $(Z_t^y, 0 \leq t \leq 1)$ on $\mathcal{C}([0, 1])$. Show that for any non-negative measurable function $F : \mathcal{C}([0, 1]) \to \mathbb{R}_+$ for $f(y) = W_0^y(F)$, we have

$$\mathbb{E}[F(B)|B_1] = f(B_1) \text{ a.s.}$$

Hint: Find a simple argument entailing that B_1 is independent of process $(B_t - tB_1, 0 \leq t \leq 1)$.

Explain why we can interpret W_0^y as the law of a Brownian motion "conditioned to hit y at time 1".

Exercise 1.3. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion in $\mathbb{R}^3$. Set $R_t = 1/|B_t|$. Show that

(i)$(R_t, t \geq 1)$ is bounded in $\mathcal{L}^2$,

(ii) $\mathbb{E}[R_t] \to 0$ as $t \to \infty$,

(iii) $(R_t)_{t>0}$ is a supermartingale.

Exercise 1.4. Fix $t \geq 0$. Show that, almost surely, Brownian motion in one dimension is not differentiable at t.

Exercise 1.5. Let $(\xi(s))_{s\leq t}$ be a standard Brownian motion in $d \geq 1$ dimensions. Set $W(t) = \cup_{s\leq t} \mathcal{B}(\xi(s), r)$, where $\mathcal{B}(x, r)$ stands for a ball centred at x of radius r, for $r > 0$.

Show that if $d = 1$, then for all t

$$\mathbb{E}[\text{vol}(W(t)] = 2r + \sqrt{\frac{8t}{\pi}}.$$

2 Poisson random measures

Exercise 2.1. Let $N, Y_n, n \in \mathbb{N}$, be independent random variables, with $N \sim P(\lambda), \lambda < \infty$ and $\mathbb{P}(Y_n = j) = p_j$, for $j = 1, \ldots, k$ and all n. Set

$$N_j = \sum_{n=1}^{N} \mathbf{1}(Y_n = j).$$

Show that $N_1, \ldots, N_k$ are independent random variables with $N_j \sim P(\lambda p_j)$ for all j.

Exercise 2.2. Let $E = \mathbb{R}_+$ and $\mu = \theta \mathbf{1}(t \geq 0)\, dt$. Let M be a Poisson random measure on $\mathbb{R}_+$ with intensity measure μ and let$(T_n)_{n\geq 1}$ and $T_0 = 0$ be a sequence of random variables such that $(T_n - T_{n-1}, n \geq 1)$ are independent exponential random variables with parameter $\theta > 0$. Show that

$$\left(N_t = \sum_{n\geq 1} \mathbf{1}(T_n \leq t),\ t \geq 0\right) \text{ and } \left(N'_t = M([0,t]),\ t \geq 0\right)$$

have the same distribution.

Exercise 2.3. Prove that the Poisson law with parameter $\lambda > 0$ is the weak limit of the Binomial law with parameters $(n, \lambda/n)$ as $n \to \infty$.

Exercise 2.4 (The bus paradox)**.** Why do we always feel we are waiting a very long time before buses arrive? This exercise gives an indication of why... well, if buses arrive according to a Poisson process.

1. Suppose buses are circulating in a city day and night since ever, the counterpart being that drivers do not officiate with a timetable. Rather, the times of arrival of buses at a given bus-stop are the atoms of a Poisson measure on $\mathbb{R}$ with intensity $\theta\, dt$, where dt is Lebesgue measure on $\mathbb{R}$. A customer arrives at a fixed time t at the bus-stop. Let S, T be the two consecutive atoms of the Poisson measure satisfying $S < t < T$. Show that the average time $\mathbb{E}[T - S]$ that elapses between the arrivals of the last bus before time t and the first bus after time t is $2/\theta$. Explain why this is twice the average time between consecutive buses. Can you see why this is so?

2. Suppose that buses start circulating at time 0, so that arrivals of buses at the station are now the jump times of a Poisson process with intensity θ on $\mathbb{R}_+$. If the customer arrives at time t, show that the average elapsed time between the bus before (time S) and after his arrival (time T) is $\theta^{-1}(2 - e^{-\theta t})$ (with the convention $S = 0$ if no atom has fallen in $[0, t]$).