Example sheet 3

1 Weak convergence

Exercise 1.1. Let \((X_n, n \geq 1)\) be a sequence of independent random variables with uniform distribution on \([0, 1]\). Let \(M_n = \max(X_1, \ldots, X_n)\). Show that \(n(1 - M_n)\) converges in distribution as \(n \to \infty\) and determine the limit law.

Exercise 1.2. Let \((X_n, n \geq 0)\) be a sequence of random variables defined on some probability space \((\Omega, \mathcal{F}, P)\) with values in a metric space \((M, d)\).

1. Suppose that \(X_n \to X_\infty\) a.s. as \(n \to \infty\). Show that \(X_n\) converges to \(X_\infty\) in distribution.

2. Suppose that \(X_n\) converges in probability to \(X_\infty\). Show that \(X_n\) converges in distribution to \(X_\infty\).

Hint: use the fact that \((X_n, n \geq 0)\) converges in probability to \(X_\infty\) if and only if for every subsequence extracted from \((X_n, n \geq 0)\), there exists a further subsequence converging a.s. to \(X_\infty\).

3. If \(X_n\) converges in distribution to a constant \(X_\infty = c\), then \(X_n\) converges in probability to \(c\).

Exercise 1.3. Suppose given sequences \((X_n, n \geq 0)\) and \((Y_n, n \geq 0)\) of real valued random variables, and two extra random variables \(X, Y\), such that \(X_n, Y_n\) respectively converge in distribution to \(X, Y\). Is it true that \((X_n, Y_n)\) converges in distribution to \((X, Y)\)? Show that this is true in the following cases:

1. For every \(n\), the random variables \(X_n\) and \(Y_n\) are independent, as well as \(X\) and \(Y\).

2. \(Y\) is a.s. constant (Hint: use 3 of the previous question).

Exercise 1.4. Let \(d \geq 1\).

1. Show that a finite family of probability measures on \(\mathbb{R}^d\) is tight.

2. Assuming Prohorov’s theorem for probability measures on \(\mathbb{R}^d\), show that if \((\mu_n, n \geq 0)\) is a sequence of non-negative measures on \(\mathbb{R}^d\) which is tight and such that

\[
\sup_{n \geq 0} \mu_n(\mathbb{R}^d) < \infty,
\]

then there exists a subsequence \(n_k\) along which \(\mu_n\) converges weakly to a limit \(\mu\).
2 Brownian motion

Exercise 2.1. Show that the standard Brownian motion in \( \mathbb{R}^d \) is the unique Gaussian process \((B_t, t \geq 0)\) with \( \mathbb{E}[B_t] = 0 \) for all \( t \geq 0 \) and \( \text{Cov}(B_s, B_t) = (s \wedge t)I_d \) for every \( s, t \geq 0 \).

Exercise 2.2. Let \( B \) be a standard Brownian in 1 dimension.

1. Show that a.s.
\[
\limsup_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = \infty \quad \text{and} \quad \liminf_{t \downarrow 0} \frac{B_t}{\sqrt{t}} = -\infty.
\]

2. Show that a.s. \( B_{n}/n \to 0 \) as \( n \to \infty \). Then show that a.s. for \( n \) large enough
\[
\sup_{t \in [n,n+1]} |B_t - B_n| \leq \sqrt{n}
\]
and conclude that \( B_t/t \to 0 \) as \( t \to \infty \) a.s.

3. Using the time inversion theorem, show that a.s.
\[
\limsup_{t \to \infty} \frac{B_t}{\sqrt{t}} = \infty \quad \text{and} \quad \liminf_{t \to \infty} \frac{B_t}{\sqrt{t}} = -\infty.
\]

Exercise 2.3. Let \( B \) be a standard Brownian motion in 1 dimension. Show that a.s. for all \( 0 < a < b < \infty \), the Brownian motion \( B \) is not monotone on the interval \([a, b]\).

Exercise 2.4. Let \((B_t, t \geq 0)\) be a standard Brownian motion in 1 dimension. Let \( T_x = \inf\{t \geq 0 : B_t = x\} \) for \( x \in \mathbb{R} \).

1. Prove that \( T_x \) has the same distribution as \((x/B_1)^2\) and compute its probability distribution function.

2. For \( x, y > 0 \), show that
\[
\mathbb{P}(T_{-y} < T_x) = \frac{x}{x+y} \quad \text{and} \quad \mathbb{E}[T_{-y} \wedge T_x] = xy.
\]

3. Show that if \( 0 < x < y \), the random variable \( T_y - T_x \) has the same law as \( T_{y-x} \) and is independent of \( \mathcal{F}_{T_x} \) (where \( (\mathcal{F}_t, t \geq 0) \) is the natural filtration of Brownian motion).

Hint: the three questions are independent.

Exercise 2.5. Let \((B_t, t \geq 0)\) be a standard Brownian motion in 1 dimension, and let \( 0 \leq a < b \).

1. Compute the mean and variance of
\[
X_n := \sum_{k=1}^{2^n} (B_{a+k(b-a)2^{-n}} - B_{a+(k-1)(b-a)2^{-n}})^2.
\]
2. Show that $X_n$ converges a.s. and give its limit.

3. Deduce that a.s. there exists no interval $[a, b]$ with $a < b$ such that $B$ is Hölder continuous with exponent $\alpha > 1/2$ on $[a, b]$, i.e. $\sup_{a \leq s, t \leq b} \left( |B_t - B_s| / |t - s|^\alpha \right) < \infty$.

**Exercise 2.6.** Let $(B_t, t \geq 0)$ be a standard Brownian motion in 1 dimension. Define $G_1 = \sup \{ t \leq 1 : B_t = 0 \}$ and $D_1 = \inf \{ t \geq 1 : B_t = 0 \}$.

1. Are these random variables stopping times? Show that $G_1$ has the same distribution as $D_1^{-1}$.

2. By applying the Markov property at time 1, compute the law of $D_1$. Deduce that of $G_1$ (this is called the arcsine law).

**Exercise 2.7.** Let $(B_t, t \geq 0)$ be a standard Brownian motion in 1 dimension. Define

$$\tau = \inf \{ t \geq 0 : B_t = \max_{0 \leq s \leq 1} B_s \}.$$ 

Is this a stopping time? *Hint:* First show that $\tau \leq 1$ a.s.