Example sheet 2

1 Discrete-time martingales

Exercise 1.1. Let $(X_n, n \geq 0)$ be a sequence of $[0,1]$-valued random variables, which satisfy the following property. First, $X_0 = a$ a.s. for some $a \in (0,1)$ and for $n \geq 0$,

$$\mathbb{P}\left(X_{n+1} = \frac{X_n}{2}\middle|\mathcal{F}_n\right) = 1 - X_n = 1 - \mathbb{P}\left(X_{n+1} = \frac{X_n+1}{2}\middle|\mathcal{F}_n\right),$$

where $\mathcal{F}_n = \sigma(X_k, 0 \leq k \leq n)$. Here, we have denoted $\mathbb{P}(A|\mathcal{G}) = \mathbb{E}[\mathbf{1}(A)|\mathcal{G}]$.

1. Prove that $(X_n, n \geq 0)$ is a martingale that converges in $\mathcal{L}^p$ for every $p \geq 1$.

2. Check that $\mathbb{E}[(X_{n+1} - X_n)^2] = \mathbb{E}[X_n(1-X_n)]/4$. Then determine $\mathbb{E}[X_\infty(1-X_\infty)]$ and deduce that law of X_∞.

Exercise 1.2. Let $(X_n, n \geq 0)$ be a martingale in $\mathcal{L}^2$. Show that its increments $(X_{n+1} - X_n : n \geq 0)$ are pairwise orthogonal, i.e. for all $n \neq m$ the increments satisfy

$$\mathbb{E}[(X_{n+1} - X_n)(X_{m+1} - X_m)] = 0.$$

Conclude that X is bounded in $\mathcal{L}^2$ if and only if

$$\sum_{n \geq 0} \mathbb{E}[(X_{n+1} - X_n)^2] < \infty.$$

Exercise 1.3 (Wald's identity). Let $(X_n, n \geq 0)$ be a sequence of independent and identically distributed real integrable random variables. We let $S_n = X_1 + \ldots + X_n$ (with $S_0 = 0$) be the associated random walk and T an $(\mathcal{F}_n)$-stopping time, where $\mathcal{F}_n = \sigma(X_k, k \leq n)$.

1. Show that if the variables X_i are non-negative, then

$$\mathbb{E}[S_T] = \mathbb{E}[T]\mathbb{E}[X_1].$$

2. Show that if $\mathbb{E}[T] < \infty$, then

$$\mathbb{E}[S_T] = \mathbb{E}[T]\mathbb{E}[X_1].$$

3. Suppose that $\mathbb{E}[X_1] = 0$ and set $T_a = \inf\{n \geq 0 : S_n \geq a\}$, for some $a > 0$. Show that $\mathbb{E}[T_a] = \infty$.

4. Suppose that $\mathbb{P}(X_1 = +1) = 2/3 = 1 - \mathbb{P}(X_1 = -1)$ and set $T_a = \inf\{n \geq 0 : S_n \geq a\}$, for some $a > 0$. Find $\mathbb{E}[T_a]$. (You cannot assume that $\mathbb{E}[T_a] < \infty$.)

Exercise 1.4 (Gambler's ruin). Suppose that $X_1, X_2, \dots$ are independent random variables with

$$\mathbb{P}(X = +1) = p,\ \mathbb{P}(X = -1) = q,$$

where $p \in (0,1)$, $q = 1 - p$ and $p \neq q$. Suppose that a and b are integers with $0 < a < b$. Define

$$S_n := a + X_1 + \cdots + X_n,\ T := \inf\{n : S_n = 0 \text{ or } S_n = b\}.$$

Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Prove that

$$M_n := \left(\frac{q}{p}\right)^{S_n} \text{ and } N_n = S_n - n(p-q)$$

define martingales M and N. Deduce the values of $\mathbb{P}(S_T = 0)$ and $\mathbb{E}[T]$.

Exercise 1.5 (Azuma–Hoeffding Inequality). (a) Show that if Y is a random variable with values in $[-c, c]$ and with $\mathbb{E}[Y] = 0$, then, for $\theta \in \mathbb{R}$,

$$\mathbb{E}[e^{\theta Y}] \leq \cosh \theta c \leq \exp\left(\frac{1}{2}\theta^2 c^2\right).$$

(b) Prove that if M is a martingale, with $M_0 = 0$ and such that for some sequence $(c_n : n \in \mathbb{N})$ of positive constants, $|M_n - M_{n-1}| \leq c_n$ for all n, then, for $x > 0$,

$$\mathbb{P}\Big(\sup_{k \leq n} M_k \geq x\Big) \leq \exp\Big(-\frac{1}{2}x^2 \Big/ \sum_{k=1}^{n} c_k^2\Big).$$

Hint for (a). Let $f(z) := \exp(\theta z)$, $z \in [-c, c]$. Then, since f is convex,

$$f(y) \leq \frac{c-y}{2c} f(-c) + \frac{c+y}{2c} f(c).$$

Hint for (b). Optimize over θ.

Exercise 1.6. Let $f : [0,1] \to \mathbb{R}$ be Lipschitz, that is, suppose that, for some $K < \infty$ and all $x, y \in [0,1]$

$$|f(x) - f(y)| \leq K|x - y|.$$

Denote by f_n the simplest piecewise linear function agreeing with f on $\{k2^{-n} : k = 0, 1, \dots, 2^n\}$. Set $M_n = f_n'$. Show that M_n converges a.e. and in $\mathcal{L}^1$ and deduce that f is the indefinite integral of a bounded function.

Exercise 1.7 (Doob's decomposition of submartingales). Let $(X_n, n \geq 0)$ be a submartingale.
1. Show that there exists a unique martingale M_n and a unique previsible process $(A_n, n \geq 0)$ (i.e. A_n is $\mathcal{F}_{n-1}$ measurable) such that $A_0 = 0$, A is increasing and $X = M + A$.
2. Show that M, A are bounded in $\mathcal{L}^1$ if and only if X is, and that $A_\infty < \infty$ a.s. in this case (and even that $\mathbb{E}[A_\infty] < \infty$), where A_∞ is the increasing limit of A_n as $n \to \infty$.

Exercise 1.8. Let $(X_n, n \geq 0)$ be a UI submartingale.
1. Show that if $X = M + A$ is the Doob decomposition of X, then M is UI.
2. Show that for every pair of stopping times S, T with $S \leq T$,

$$\mathbb{E}[X_T | \mathcal{F}_S] \geq X_S.$$

2 Continuous-time processes

Exercise 2.1 (Gaussian processes). A real-valued process $(X_t, t \geq 0)$ is called a *Gaussian process* if for every $t_1 < t_2 < \ldots < t_k$, the random vector $(X_{t_1}, \ldots, X_{t_k})$ is a Gaussian random vector. Show that the law of a Gaussian process is uniquely characterized by the numbers $\mathbb{E}[X_t], t \geq 0$ and $\mathrm{Cov}(X_s, X_t)$ for $s, t \geq 0$.

Exercise 2.2. Let $T \sim E(\lambda)$. Define

$$Z_t = \begin{cases} 0 & \text{if } t < T \\ 1 & \text{if } t \geq T \end{cases}, \quad \mathcal{F}_t = \sigma\{Z_s : s \leq t\}, \; M_t = \begin{cases} 1 - e^{\lambda t} & \text{if } t < T \\ 1 & \text{if } t \geq T. \end{cases}$$

Prove that $\mathbb{E}[|M_t|] < \infty$, and that $\mathbb{E}[M_t; \{T > r\}] = \mathbb{E}[M_s; \{T > r\}]$ for $r \leq s \leq t$, and hence deduce that M_t is a cadlag martingale with respect to the filtration $\{\mathcal{F}_t\}$.

Is M bounded in $\mathcal{L}^1$? Is M uniformly integrable? Is M_{T-} in $\mathcal{L}^1$?

Exercise 2.3. Let T be a random variable with values in $(0, \infty)$ and with strictly positive continuous density f on $(0, \infty)$ and distribution function $F(t) = \mathbb{P}(T \leq t)$. Define

$$A_t = \int_0^t \frac{f(s)}{1 - F(s)} \, ds, \quad 0 \leq t < \infty.$$

By expressing the distribution function of $A_T, G(t) = \mathbb{P}(A_T \leq t)$, in terms of the inverse function A^{-1} of A, or otherwise, deduce that A_T has the exponential distribution of mean 1.

Define Z_t and $\mathcal{F}_t$ as in Exercise 2.2 above, and prove that $M_t = Z_t - A_{t \wedge T}$ is a cadlag martingale relative to $\{\mathcal{F}_t\}$. The function A_t is called the *hazard function* for T.