Example sheet 2

1 Discrete-time martingales

Exercise 1.1 (Polya's urn). At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn and is replaced together with a new ball of the same colour. Just after time n, there are therefore $n + 2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let $M_n = (B_n + 1)/(n + 2)$ the proportion of black balls in the urn just after time n. Prove that, relative to a natural filtration which you should specify, M is a martingale. Show that it converges a.s. and in $\mathcal{L}^p$ for all $p \geq 1$ to a $[0, 1]$-valued random variable X_∞.

Show that for every k, the process

$$\frac{(B_n + 1)(B_n + 2) \ldots (B_n + k)}{(n + 2)(n + 3) \ldots (n + k + 1)}, \; n \geq 1$$

is a martingale. Deduce the value of $\mathbb{E}[X_\infty^k]$, and finally the law of X_∞.

Reobtain this result by showing directly that $\mathbb{P}(B_n = k) = (n + 1)^{-1}$ for $0 \leq k \leq n$.

Prove that for $0 < \theta < 1$, $(N_n(\theta))_{n \geq 0}$ is a martingale, where

$$N_n(\theta) := \frac{(n + 1)!}{B_n!(n - B_n)!} \, \theta^{B_n} (1 - \theta)^{n - B_n}.$$

Exercise 1.2 (Bayes' urn). A random number Θ is chosen uniformly between 0 and 1, and a coin with probability Θ of heads is minted. The coin is tossed repeatedly. Let B_n be the number of heads in n tosses. Prove that (B_n) has exactly the same probabilistic structure as the (B_n) sequence in Exercise 1.1. Prove that $N_n(\theta)$ is a conditional density function of Θ given $B_1, B_2, \ldots, B_n$.

Exercise 1.3 (ABRACADABRA). At each of times $1, 2, 3, \ldots$, a monkey types a capital letter at random, the sequence of letters typed forming a sequence of independent random variables, each chosen uniformly from amongst the 26 possible capital letters.

Just before each time $n = 1, 2, \ldots$, a new gambler arrives on the scene. He bets \$1 that

the n^{th} letter will be A.

If he loses, he leaves. If he wins, he receives \$26 all of which he bets on the event that

the $(n + 1)^{\text{th}}$ letter will be B.

If he loses, he leaves. If he wins, he bets his whole current fortune $\$26^2$ that

$$\text{the } (n+2)^{\text{th}} \text{ letter will be } R$$

and so on through the ABRACADABRA sequence. Let T be the first time by which the monkey has produced the consecutive sequence ABRACADABRA. Prove, by a martingale argument, that

$$\mathbb{E}[T] = 26^{11} + 26^4 + 26.$$

Exercise 1.4. Let $(X_n, n \geq 0)$ be a sequence of $[0,1]$-valued random variables, which satisfy the following property. First, $X_0 = a$ a.s. for some $a \in (0,1)$ and for $n \geq 0$,

$$\mathbb{P}\left(X_{n+1} = \frac{X_n}{2} \middle| \mathcal{F}_n\right) = 1 - X_n = 1 - \mathbb{P}\left(X_{n+1} = \frac{X_n + 1}{2} \middle| \mathcal{F}_n\right),$$

where $\mathcal{F}_n = \sigma(X_k, 0 \leq k \leq n)$. Here, we have denoted $\mathbb{P}(A|\mathcal{G}) = \mathbb{E}[\mathbf{1}(A)|\mathcal{G}]$.

1. Prove that $(X_n, n \geq 0)$ is a martingale that converges in $\mathcal{L}^p$ for every $p \geq 1$.

2. Check that $\mathbb{E}[(X_{n+1} - X_n)^2] = \mathbb{E}[X_n(1 - X_n)]/4$. Then determine $\mathbb{E}[X_\infty(1 - X_\infty)]$ and deduce that law of X_∞.

Exercise 1.5. Let $(X_n, n \geq 0)$ be a martingale in $\mathcal{L}^2$. Show that its increments $(X_{n+1} - X_n : n \geq 0)$ are pairwise orthogonal, i.e. for all $n \neq m$ the increments satisfy

$$\mathbb{E}[(X_{n+1} - X_n)(X_{m+1} - X_m)] = 0.$$

Conclude that X is bounded in $\mathcal{L}^2$ if and only if

$$\sum_{n \geq 0} \mathbb{E}[(X_{n+1} - X_n)^2] < \infty.$$

Exercise 1.6 (Wald's identity). Let $(X_n, n \geq 0)$ be a sequence of independent and identically distributed real integrable random variables. We let $S_n = X_1 + \ldots + X_n$ (with $S_0 = 0$) be the associated random walk and T an $(\mathcal{F}_n)$-stopping time, where $\mathcal{F}_n = \sigma(X_k, k \leq n)$.

1. Show that if the variables X_i are non-negative, then

$$\mathbb{E}[S_T] = \mathbb{E}[T]\mathbb{E}[X_1].$$

2. Show that if $\mathbb{E}[T] < \infty$, then

$$\mathbb{E}[S_T] = \mathbb{E}[T]\mathbb{E}[X_1].$$

3. Suppose that $\mathbb{E}[X_1] = 0$ and set $T_a = \inf\{n \geq 0 : S_n \geq a\}$, for some $a > 0$. Show that $\mathbb{E}[T_a] = \infty$.

4. Suppose that $\mathbb{P}(X_1 = +1) = 2/3 = 1 - \mathbb{P}(X_1 = -1)$ and set $T_a = \inf\{n \geq 0 : S_n \geq a\}$, for some $a > 0$. Find $\mathbb{E}[T_a]$. (You cannot assume that $\mathbb{E}[T_a] < \infty$.)

Exercise 1.7 (**Gambler's ruin**). Suppose that $X_1, X_2, \dots$ are independent random variables with

$$\mathbb{P}(X = +1) = p, \ \mathbb{P}(X = -1) = q,$$

where $p \in (0,1)$, $q = 1 - p$ and $p \neq q$. Suppose that a and b are integers with $0 < a < b$. Define

$$S_n := a + X_1 + \cdots + X_n, \ T := \inf\{n : S_n = 0 \text{ or } S_n = b\}.$$

Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Prove that

$$M_n := \left(\frac{q}{p}\right)^{S_n} \text{ and } N_n = S_n - n(p - q)$$

define martingales M and N. Deduce the values of $\mathbb{P}(S_T = 0)$ and $\mathbb{E}[T]$.

Exercise 1.8 (**Azuma–Hoeffding Inequality**). (a) Show that if Y is a random variable with values in $[-c, c]$ and with $\mathbb{E}[Y] = 0$, then, for $\theta \in \mathbb{R}$,

$$\mathbb{E}[e^{\theta Y}] \leq \cosh \theta c \leq \exp\left(\frac{1}{2}\theta^2 c^2\right).$$

(b) Prove that if M is a martingale, with $M_0 = 0$ and such that for some sequence $(c_n : n \in \mathbb{N})$ of positive constants, $|M_n - M_{n-1}| \leq c_n$ for all n, then, for $x > 0$,

$$\mathbb{P}\Big(\sup_{k \leq n} M_k \geq x\Big) \leq \exp\Big(-\frac{1}{2}x^2 \Big/ \sum_{k=1}^{n} c_k^2\Big).$$

Hint for (a). Let $f(z) := \exp(\theta z)$, $z \in [-c, c]$. Then, since f is convex,

$$f(y) \leq \frac{c - y}{2c} f(-c) + \frac{c + y}{2c} f(c).$$

Hint for (b). Optimize over θ.

Exercise 1.9. Let $f : [0,1] \to \mathbb{R}$ be Lipschitz, that is, suppose that, for some $K < \infty$ and all $x, y \in [0,1]$

$$|f(x) - f(y)| \leq K|x - y|.$$

Denote by f_n the simplest piecewise linear function agreeing with f on $\{k2^{-n} : k = 0, 1, \dots, 2^n\}$. Set $M_n = f_n'$. Show that M_n converges a.e. and in $\mathcal{L}^1$ and deduce that f is the indefinite integral of a bounded function.

Exercise 1.10 (**Doob's decomposition of submartingales**). Let $(X_n, n \geq 0)$ be a submartingale.

1. Show that there exists a unique martingale M_n and a unique previsible process $(A_n, n \geq 0)$ (i.e. A_n is $\mathcal{F}_{n-1}$ measurable) such that $A_0 = 0$, A is increasing and $X = M + A$.
2. Show that M, A are bounded in $\mathcal{L}^1$ if and only if X is, and that $A_\infty < \infty$ a.s. in this case (and even that $\mathbb{E}[A_\infty] < \infty$), where A_∞ is the increasing limit of A_n as $n \to \infty$.

Exercise 1.11. Let $(X_n, n \geq 0)$ be a UI submartingale.

1. Show that if $X = M + A$ is the Doob decomposition of X, then M is UI.
2. Show that for every pair of stopping times S, T with $S \leq T$,

$$\mathbb{E}[X_T | \mathcal{F}_S] \geq X_S.$$

2 Weak convergence

Exercise 2.1. Let $(X_n, n \geq 1)$ be a sequence of independent random variables with uniform distribution on $[0, 1]$. Let $M_n = \max(X_1, \dots, X_n)$. Show that $n(1 - M_n)$ converges in distribution as $n \to \infty$ and determine the limit law.

Exercise 2.2. Let $(X_n, n \geq 0)$ be a sequence of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a metric space (M, d).

1. Suppose that $X_n \to X_\infty$ a.s. as $n \to \infty$. Show that X_n converges to X_∞ in distribution.

2. Suppose that X_n converges in probability to X_∞. Show that X_n converges in distribution to X_∞.
Hint: use the fact that $(X_n, n \geq 0)$ converges in probability to X_∞ if and only if for every subsequence extracted from $(X_n, n \geq 0)$, there exists a further subsequence converging a.s. to X_∞.

3. If X_n converges in distribution to a constant $X_\infty = c$, then X_n converges in probability to c.

Exercise 2.3. Suppose given sequences $(X_n, n \geq 0)$ and $(Y_n, n \geq 0)$ of real valued random variables, and two extra random variables X, Y, such that X_n, Y_n respectively converge in distribution to X, Y. Is it true that (X_n, Y_n) converges in distribution to (X, Y)? Show that this is true in the following cases:
1. For every n, the random variables X_n and Y_n are independent, as well as X and Y.

2. Y is a.s. constant (*Hint:* use 3 of the previous question).

Exercise 2.4. Let $d \geq 1$.

1. Show that a finite family of probability measures on $\mathbb{R}^d$ is tight.

2. Assuming Prohorov's theorem for probability measures on $\mathbb{R}^d$, show that if $(\mu_n, n \geq 0)$ is a sequence of non-negative measures on $\mathbb{R}^d$ which is tight and such that

$$\sup_{n \geq 0} \mu_n(\mathbb{R}^d) < \infty,$$

then there exists a subsequence n_k along which μ_n converges weakly to a limit μ.