Example sheet 1

1 Conditional expectation

Exercise 1.1. Let X and Y be integrable random variables and suppose that

$$\mathbb{E}[X|Y] = Y \quad \text{and} \quad \mathbb{E}[Y|X] = X \quad \text{a.s.}$$

Show that $X = Y$ a.s.

Hint: Consider quantities like $\mathbb{E}[(X-Y)\mathbb{1}(X > c, Y \leq c)] + \mathbb{E}[(X-Y)\mathbf{1}(X \leq c, Y \leq c)]$.

Exercise 1.2. Let X, Y be two independent Bernoulli random variables with parameter $p \in (0,1)$. Let $Z = \mathbf{1}(X+Y=0)$. Compute $\mathbb{E}[X|Z]$ and $\mathbb{E}[Y|Z]$.

Exercise 1.3. Let X, Y be two independent exponential random variables of parameter θ. Let $Z = X + Y$, then check that the distribution of Z is gamma with parameter $(2,\theta)$, whose density with respect to the Lebesgue measure is $\theta^2 x e^{-\theta x}\mathbf{1}(x \geq 0)$. Show that for any non-negative measurable h,

$$\mathbb{E}[h(X)|Z] = \frac{1}{Z}\int_0^Z h(u)\,du.$$

Conversely, let Z be a random variable with a $\Gamma(2,\theta)$ distribution, and suppose that X is a random variable whose conditional distribution given Z is uniform on $[0, Z]$. Namely, for every Borel non-negative function h

$$\mathbb{E}[h(X)|Z] = \frac{1}{Z}\int_0^Z h(u)\,du \text{ a.s.}$$

Show that X and $Z - X$ are independent, with exponential law.

Exercise 1.4. Let $X \geq 0$ be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra.

1. Show that $X > 0$ implies that $\mathbb{E}[X|\mathcal{G}] > 0$ up to an event of zero probability.

2. Show that $\{\mathbb{E}[X|\mathcal{G}] > 0\}$ is the smallest $\mathcal{G}$-measurable event that contains the event $\{X > 0\}$ up to zero probability events.

Exercise 1.5. Suppose given $a, b > 0$, and let X, Y be two random variables with values in $\mathbb{Z}_+$ and $\mathbb{R}_+$ respectively, whose distribution is given by the formula

$$\mathbb{P}(X = n, Y \leq t) = b \int_0^t \frac{(ay)^n}{n!} \exp(-(a+b)y)\, dy.$$

Let $n \in \mathbb{Z}_+$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function, compute $\mathbb{E}[h(Y)|X = n]$. Then compute $\mathbb{E}[Y/(X+1)], \mathbb{E}[\mathbf{1}(X = n)|Y]$ and $\mathbb{E}[X|Y]$.

Exercise 1.6 (Conditional independence). Let $\mathcal{G} \subseteq \mathcal{F}$ be a sub-σ-algebra. Two random variables X, Y are said to be independent conditionally on $\mathcal{G}$ if for every non-negative measurable f, g,

$$\mathbb{E}[f(X)g(Y)|\mathcal{G}] = \mathbb{E}[f(X)|\mathcal{G}]\mathbb{E}[g(Y)|\mathcal{G}] \text{ a.s.}$$

What are two random variables independent conditionally on $\{\varnothing, \Omega\}$? On $\mathcal{F}$?

Show that X, Y are independent conditionally on $\mathcal{G}$ if and only if for every non-negative $\mathcal{G}$-measurable random variable Z, and every f, g non-negative measurable functions,

$$\mathbb{E}[f(X)g(Y)Z] = \mathbb{E}[f(X)Z\mathbb{E}[g(Y)|\mathcal{G}]],$$

and this if and only for every measurable non-negative g,

$$\mathbb{E}[g(Y)|\mathcal{G} \vee \sigma(X)] = \mathbb{E}[g(Y)|\mathcal{G}].$$

Exercise 1.7. Give an example of a random variable X and two σ-algebras $\mathcal{H}$ and $\mathcal{G}$ such that X is independent of $\mathcal{H}$ and $\mathcal{G}$ is independent of $\mathcal{H}$, nevertheless

$$\mathbb{E}[X|\sigma(\mathcal{G}, \mathcal{H})] \neq \mathbb{E}[X|\mathcal{G}].$$

Hint: Consider coin tosses.

2 Discrete-time martingales

Exercise 2.1. Let $(X_n, n \geq 0)$ be an integrable process with values in a countable subset $E \subset \mathbb{R}$. Show that X is a martingale with respect to its natural filtration if and only if for every n and every $i_0, \dots, i_n \in E$, we have

$$\mathbb{E}[X_{n+1}|X_0 = i_0, \dots, X_n = i_n] = i_n.$$

Exercise 2.2. A process $C = (C_n, n \geq 0)$ is called previsible, if C_n is $\mathcal{F}_{n-1}$-measurable, for all $n \geq 1$. Let C be a previsible process and X a martingale (resp. supermartingale). We set

$$Y_n = \sum_{k \leq n} C_k(X_k - X_{k-1}), \text{ for all } n \geq 0.$$

Show that if C is bounded then $(Y_n, n \geq 0)$ is a martingale (if $C_n \geq 0$ for all n and bounded then it is a supermartingale).

We write $Y_n = (C \bullet X)_n$ and call it the martingale transform of X by C. It is the discrete analogue of the stochastic integral $\int C\, dX$. More on that in the "Stochastic calculus" course next term.

Exercise 2.3. Let $(X_n, n \geq 1)$ be a sequence of independent random variables with respective laws given by

$$\mathbb{P}(X_n = -n^2) = \frac{1}{n^2} \quad \text{and} \quad \mathbb{P}\left(X_n = \frac{n^2}{n^2-1}\right) = 1 - \frac{1}{n^2}.$$

Let $S_n = X_1 + \ldots + X_n$. Show that $S_n/n \to 1$ a.s. as $n \to \infty$ and deduce that $(S_n, n \geq 0)$ is a martingale which converges to $+\infty$.

Exercise 2.4. Let $(\Omega, \mathcal{F}, (\mathcal{F}_n), \mathbb{P})$ be a filtered probability space. Let $A \in \mathcal{F}_n$ for some n and let $m, m' \geq n$. Show that $m\mathbf{1}(A) + m'\mathbf{1}(A^c)$ is a stopping time.

Show that an adapted process $(X_n, n \geq 0)$ with respect to some filtered probability space is a martingale if and only if it is integrable, and for every bounded stopping time T, $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Exercise 2.5. Let X be a martingale (resp. supermartingale) on some filtered probability space, and let T be an a.s. finite stopping time. Prove that $\mathbb{E}[X_T] = \mathbb{E}[X_0]$ (resp. $\mathbb{E}[X_T] \leq \mathbb{E}[X_0]$) if either one of the following conditions holds:

1. X is bounded ($\exists M > 0 : \ \forall n \geq 0, \ |X_n| \leq M$ a.s.)
2. X has bounded increments ($\exists M > 0 : \ \forall n \geq 0, |X_{n+1} - X_n| \leq M$ a.s.) and $\mathbb{E}[T] < \infty$.

Exercise 2.6. Let T be an $(\mathcal{F}_n, n \geq 0)$-stopping time such that for some integer $N > 0$ and $\varepsilon > 0$,

$$\mathbb{P}(T \leq N + n | \mathcal{F}_n) \geq \varepsilon, \text{ for every } n \geq 0.$$

Show that $\mathbb{E}[T] < \infty$.
Hint: Find bounds for $\mathbb{P}(T > kN)$.

Exercise 2.7. Your winnings per unit stake on game n are ε_n, where the ε_n are independent random variables with

$$\mathbb{P}(\varepsilon_n = 1) = p \quad \text{and} \quad \mathbb{P}(\varepsilon_n = -1) = q,$$

where $p \in (1/2, 1)$ and $q = 1 - p$. Your stake C_n on game n must lie between 0 and Z_{n-1}, where Z_{n-1} is your fortune at time $n - 1$. Your object is to maximize the expected 'interest rate' $\mathbb{E}[\log(Z_N/Z_0)]$, where N is a given integer representing the length of the game, and Z_0, your fortune at time 0, is a given constant. Let $\mathcal{F}_n = \sigma(\varepsilon_1, \ldots, \varepsilon_n)$. Show that if C is any *previsible* strategy, that is C_n is $\mathcal{F}_{n-1}$-measurable for all n, then $\log Z_n - n\alpha$ is a supermartingale, where α denotes the *entropy*

$$\alpha = p \log p + q \log q + \log 2,$$

so that $\mathbb{E}[\log(Z_n/Z_0)] \leq N\alpha$, but that, for a certain strategy, $\log Z_n - n\alpha$ is a *martingale*. What is the best strategy?

Exercise 2.8 (Polya's urn). At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \ldots$, a ball is chosen at random from the urn and is replaced together with a new ball of the same colour. Just after time n, there are therefore $n + 2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n. Let

$M_n = (B_n + 1)/(n + 2)$ the proportion of black balls in the urn just after time n. Prove that, relative to a natural filtration which you should specify, M is a martingale. Show that it converges a.s. and in $\mathcal{L}^p$ for all $p \geq 1$ to a $[0, 1]$-valued random variable X_∞.

Show that for every k, the process

$$\frac{(B_n + 1)(B_n + 2) \dots (B_n + k)}{(n + 2)(n + 3) \dots (n + k + 1)}, \; n \geq 1$$

is a martingale. Deduce the value of $\mathbb{E}[X_\infty^k]$, and finally the law of X_∞.

Reobtain this result by showing directly that $\mathbb{P}(B_n = k) = (n + 1)^{-1}$ for $0 \leq k \leq n$.

Prove that for $0 < \theta < 1$, $(N_n(\theta))_{n \geq 0}$ is a martingale, where

$$N_n(\theta) := \frac{(n + 1)!}{B_n!(n - B_n)!} \theta^{B_n} (1 - \theta)^{n - B_n}.$$

Exercise 2.9 (Bayes' urn). A random number Θ is chosen uniformly between 0 and 1, and a coin with probability Θ of heads is minted. The coin is tossed repeatedly. Let B_n be the number of heads in n tosses. Prove that (B_n) has exactly the same probabilistic structure as the (B_n) sequence in Exercise 2.8. Prove that $N_n(\theta)$ is a conditional density function of Θ given $B_1, B_2, \dots, B_n$.

Exercise 2.10 (ABRACADABRA). At each of times $1, 2, 3, \dots$, a monkey types a capital letter at random, the sequence of letters typed forming a sequence of independent random variables, each chosen uniformly from amongst the 26 possible capital letters.

Just before each time $n = 1, 2, \dots$, a new gambler arrives on the scene. He bets \$1 that

the n^{th} letter will be A.

If he loses, he leaves. If he wins, he receives \$26 all of which he bets on the event that

the $(n + 1)^{\text{th}}$ letter will be B.

If he loses, he leaves. If he wins, he bets his whole current fortune $\$26^2$ that

the $(n + 2)^{\text{th}}$ letter will be R

and so on through the ABRACADABRA sequence. Let T be the first time by which the monkey has produced the consecutive sequence ABRACADABRA. Prove, by a martingale argument, that

$$\mathbb{E}[T] = 26^{11} + 26^4 + 26.$$