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Example Sheet 2

**1.** Let $X$ be a reversible Markov chain on a finite state space $E$ with transition matrix $P$ and invariant distribution $\pi$. Prove a generalisation of the Poincaré inequality, i.e. for all $f : E \to \mathbb{R}$ show that

$$\mathrm{Var}_\pi\left(P^t f\right) \le e^{-2t/t_{\mathrm{rel}}}\mathrm{Var}_\pi\left(f\right).$$

**2.** Let $P$ be a reversible transition matrix on a finite state space with invariant distribution $\pi$. Define the total variation distance from stationarity from a typical point, i.e. for all $t$

$$d_{\mathrm{ave}}(t) = \sum_x \pi(x)\left\|P^t(x,\cdot) - \pi\right\|_{\mathrm{TV}}.$$

Suppose that $1 = \lambda_1 \ge \ldots \ge \lambda_n \ge -1$ are the eigenvalues, then show that

$$4d_{\mathrm{ave}}(t)^2 \le \sum_{j=2}^n \lambda_j^{2t}.$$

**3.** Consider a lazy simple random walk on $\mathbb{Z}_n$. Show that for all $\alpha > 0$ we have as $n \to \infty$

$$d(\alpha n^2) \to \int_0^1 \left|\sum_{k=1}^\infty e^{-\alpha\pi^2k^2}\cos(2\pi ku)\right| du.$$

(*Hint*: First write

$$d(\alpha n^2) = \frac{1}{2}\int_0^1 \left|1 - nP^{\lceil \alpha n^2\rceil}(0, \lfloor un\rfloor)\right| du$$

and then use the spectral theorem.)

**4.** Let $X$ be an irreducible Markov chain on the finite state space $E$ with transition matrix $P$ and invariant distribution $\pi$.

(i) Define the separation distance $s(t) = \max_{x,y}\left(1 - P^t(x,y)/\pi(y)\right)$. Show that $s(t)$ is decreasing as a function of $t$.

(ii) Define $t_{\mathrm{sep}}(\varepsilon) = \min\{t \ge 0 : s(t) \le \varepsilon\}$. Show that for all $\varepsilon \in (0,1]$ and all $k \in \mathbb{N}$ we have that

$$t_{\mathrm{sep}}(\varepsilon^k) \le kt_{\mathrm{sep}}(\varepsilon).$$

**5.** Consider two copies $K_n$ and $K_n'$ of the complete graph joined by a single edge. Find the order of the mixing time for a lazy simple random walk on the resulting graph.

**6.** Let $X$ be an irreducible, lazy and reversible Markov chain on a finite state space with transition matrix $P$ and stationary distribution $\pi$. Let $1 = \lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$ be the eigenvalues of $P$.

(i) Show that

$$\mathbb{E}_\pi[\tau_\pi] := \sum_{x,y}\pi(x)\pi(y)\mathbb{E}_x[\tau_y] = \sum_{i\ge 2}\frac{1}{1-\lambda_i}.$$

(*Hint:* Use question 12(b) from the first example sheet.)

(ii) Show that

$$\sum_{t=k}^{\infty}(P^t(x,x)-\pi(x)) \le e^{-k/t_{\text{rel}}}\mathbb{E}_\pi[\tau_x]\,.$$

**7.** Let $X$ be a reversible Markov chain on the finite state space $E$ with transition matrix $P$ and invariant distribution $\pi$.

(i) Prove that for all $x,y$

$$\frac{P^{2t}(x,y)}{\pi(y)} \ge \left(1-\max_{z,w}\left\|P^t(z,\cdot)-P^t(w,\cdot)\right\|_{\text{TV}}\right)^2.$$

Deduce that

$$P^{2t_{\text{mix}}}(x,y) \ge \frac{1}{4}\pi(y)$$

and that there exists a transition matrix $\widetilde{P}$ such that

$$P^{2t_{\text{mix}}}(x,y) = \frac{1}{4}\pi(y)+\frac{3}{4}\widetilde{P}(x,y)$$

(ii) Define

$$t_{\text{stop}} = \max_x \min\{\mathbb{E}_x[\Lambda_x] : \Lambda_x \text{ is a randomised stopping time s.t. } \mathbb{P}_x(X_{\Lambda_x}\in\cdot)=\pi(\cdot)\}.$$

(It is not clear by the definition that a stationary time achieving the minimum exists. One such example is the filling rule introduced by Baxter and Chacon.) By defining an appropriate stationary time, prove that

$$t_{\text{stop}} \le 8t_{\text{mix}}.$$

We say that a randomised stopping time $T$ starting from $x$ has a halting state if there exists $z\in E$ such that $T\le\tau_z$, where $\tau_z=\min\{t\ge 0: X_t=z\}$.

(*Harder*) Show that if $T$ has a halting state, then it is mean optimal, in the sense that

$$\mathbb{E}_x[T] = \min\{\mathbb{E}_x[\Lambda_x] : \Lambda_x \text{ is a randomised stopping time s.t. } \mathbb{P}_x(X_{\Lambda_x}\in\cdot)=\pi(\cdot)\}.$$

(*Hint*: For a stopping time $S$ consider the exit frequencies from each state, i.e. $\nu(y) = \mathbb{E}_x\left[\sum_{k=0}^{T-1}\mathbf{1}(X_k=y)\right]$ for all $y$ and compare them for different stopping times. Then use the uniqueness of the invariant measure up to multiplying by a constant.)

**8.** Let $X$ be a reversible Markov chain with values in the finite space $E$, transition matrix $P$ and invariant distribution $\pi$.

(a) Let $\varphi$ be an eigenfunction of $P$ corresponding to eigenvalue $\lambda\neq 1$ and $\|\varphi\|_2=1$. Show that

$$\mathbb{E}_\pi\left[\left(\sum_{s=0}^{t-1}\varphi(X_s)\right)^2\right] \le \frac{2t}{1-\lambda}.$$

(b) Let $f:E\to\mathbb{R}$ be a function with $\mathbb{E}_\pi[f]=0$. Recall $\gamma=1-\lambda_2$ is the spectral gap. Show that

$$\mathbb{E}_\pi\left[\left(\sum_{s=0}^{t-1}f(X_s)\right)^2\right] \le \frac{2t\mathbb{E}_\pi\left[f^2\right]}{\gamma}.$$

(c) Using coupling or otherwise, show that if $r \geq t_{\mathrm{mix}}(\varepsilon/2)$ and $t \geq 4t_{\mathrm{rel}}\mathrm{Var}_\pi(f)/(\eta^2\varepsilon)$, then for all $x \in E$

$$\mathbb{P}_x\left(\left|\frac{1}{t}\sum_{s=0}^{t-1} f(X_{r+s}) - \mathbb{E}_\pi[f]\right| \geq \eta\right) \leq \varepsilon.$$

**9**. Let $X$ be a reversible Markov chain on a finite state space $E$ with transition matrix $P$ and invariant distribution $\pi$. Let $A \subsetneq E$ and let $B = A^c$ with $k = |B|$. Suppose that the sub-stochastic matrix $P_B$ (the restriction of $P$ to $B$, i.e. $P_B(x,y) = P(x, y$ for $x, y \in B))$ is irreducible, in the sense that for all $x, y \in B$, there exists $n \geq 0$ such that $P_B^n(x,y) > 0$.

(i) By defining an appropriate inner product, show that $P_B$ has $k$ real eigenvalues

$$1 \geq \gamma_1 > \gamma_2 \geq \ldots \geq \gamma_k.$$

(ii) Show that there exist nonnegative numbers $a_1, \ldots, a_k$ satisfying $\sum_i a_i = 1$ such that for all $t \geq 0$ we have

$$\mathbb{P}_{\pi_B}(\tau_A > t) = \sum_{i=1}^k a_i \gamma_i^t,$$

where $\pi_B(x) = \pi(x)/\pi(B)$ for all $x \in B$.

(iii) The Perron Frobenius theorem gives that $\gamma_1 > 0$ and $\gamma_1 \geq -\gamma_k$. Using the Courant-Fischer characterisation of eigenvalues establish that

$$\gamma_1 \leq 1 - \frac{\pi(A)}{t_{\mathrm{rel}}}.$$

(iv) Deduce that $\mathbb{P}_{\pi_B}(\tau_A > t) \leq \left(1 - \frac{\pi(A)}{t_{\mathrm{rel}}}\right)^t \leq \exp\left(-\frac{t\pi(A)}{t_{\mathrm{rel}}}\right)$.

(v) By the Perron Frobenius theorem the left eigenvector $v$ corresponding to $\gamma_1 > 0$ is strictly positive. Let $\alpha$ be a probability distribution given by $\alpha = v/\sum_i v(i)$. Show that when the starting distribution is $\alpha$, then the law of $\tau_A$ is geometric with parameter $\gamma_1$.

Prove that for all $t$ and all $y$

$$\mathbb{P}_\alpha(X_t = y \mid \tau_A > t) = \alpha(y).$$

Finally show that if $P_B$ is in addition aperiodic, then for all $x \notin A$ we have

$$\mathbb{P}_x(X_t = y \mid \tau_A > t) \to \alpha(y) \text{ as } t \to \infty.$$

(The distribution $\alpha$ is called the quasi-stationary distribution.)