Brownian motion with variable drift

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joint work with Yuval Peres

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Theorem (Lévy 1940)

Let $B$ be a planar Brownian motion. Then $L(B_{[0,1]}) = 0$ a.s.
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$$\mathcal{L}(B[0,1]) = 0 \text{ a.s.}$$
Question

Let $f$ be a continuous function. Does $(B + f)[0, 1]$ still have 0 area?
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*Let f be a continuous function. Does \((B + f)[0, 1]\) still have 0 area?*

**An a.s. property insensitive to the drift:**
For any \(f\) continuous, \(B + f\) is nowhere differentiable a.s.
Denote by $D[0, 1]$ the **Dirichlet space**

$$D[0, 1] = \left\{ f \in C[0, 1] : \exists g \in L^2[0, 1] \text{ s.t. } f(t) = \int_0^t g(s)ds, \forall t \in [0, 1] \right\}.$$
Cameron–Martin Theorem

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**Theorem (Cameron–Martin 1944)**

*If* $f \in D[0, 1]$, *then the law of* $B$ *is mutually absolutely continuous w.r.t. the law of* $B + f$. 
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**Theorem (Cameron–Martin 1944)**

*If $f \in D[0, 1]$, then the law of $B$ is mutually absolutely continuous w.r.t. the law of $B + f$.***

Hence, if $f \in D[0, 1]$, then $\mathcal{L}(B + f)[0, 1] = 0$ a.s.
Graversen’s result

Theorem (Graversen 1982)

For all $0 < \alpha < 1/2$, there exists a Hölder($\alpha$) continuous function $f : \mathbb{R}_+ \to \mathbb{R}^2$ s.t. $\mathbb{E}[\mathcal{L}(B + f)[0, 1]] > 0$. 

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Le-Gall’s result

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**Theorem (Le-Gall 1988)**

*If $f$ is Hölder(1/2), then*

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We will see: same transition from Hölder($\alpha$) for $\alpha < 1/2$ to $\alpha = 1/2$ applies to a large variety of properties of Brownian motion.
Very recently, Antunović, Peres and Vermesi strengthened Graversen’s result and they proved

\[
\text{Theorem (Antunović et al. 2010)} \quad \text{For any } \alpha < \frac{1}{2}, \text{ there exists a } \text{Hölder}(\alpha) \text{ function } f : \mathbb{R}^+ \to \mathbb{R}^2 \text{ for which } (B + f)[0,1] \text{ completely covers an open set a.s.}
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In all these works it was not clear whether for any continuous $f$

$$\mathbb{P}(\mathcal{L}(B + f)[0, 1] > 0) \in \{0, 1\}.$$
A remaining question

In all these works it was not clear whether for any continuous $f$

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This was the impetus for our work.
Let \((B_t, 0 \leq t \leq 1)\) be a standard Brownian motion in \(\mathbb{R}^d\) and let \(f : [0, 1] \rightarrow \mathbb{R}^d\) be a continuous function.
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**Theorem (Peres and S.)**

Let \(\mathbb{P}(L(B_t + f)[0, 1] > 0)\) be a standard Brownian motion in \(\mathbb{R}^d\) and let \(f : [0, 1] \to \mathbb{R}^d\) be a continuous function.
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**Theorem (Peres and S.)**

- \(\mathbb{P}(\mathcal{L}(B + f)[0, 1] > 0) \in \{0, 1\}\).
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\]

\[
\dim(B + f)[0, 1] = c \text{ a.s., where } c \text{ is a positive constant and } \dim \text{ is the Hausdorff dimension}.
\]
Beyond the Cameron–Martin theorem

Again the same setting, $B$ is a standard Brownian motion and $D[0, 1]$ is the Dirichlet space

$$D[0, 1] = \left\{ f \in C[0, 1] : \exists g \in L^2[0, 1] \text{ s.t. } f(t) = \int_0^t g(s)ds, \forall t \in [0, 1] \right\}.$$
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**Theorem (Cameron–Martin 1944)**

If $f \notin D[0,1]$, then the law of $B$ and the law of $B + f$ are singular.
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**Theorem (Cameron–Martin 1944)**

If $f \notin D[0, 1]$, then the law of $B$ and the law of $B + f$ are singular.

As a consequence, when $f \notin D[0, 1]$, there is some a.s. property of Brownian motion that fails for $B + f$. 

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Cauchy–Schwartz inequality gives that if $f \in D[0, 1]$, then $f$ is Hölder($1/2$).
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The space of Hölder($\alpha$) continuous functions is much larger than $D[0,1]$. Indeed, for any $\alpha \in (0,1/2]$, most Hölder($\alpha$) continuous functions are nowhere differentiable.
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**Question**

*Does \( B + f \) hit the same sets as \( B \), if \( f \) is Hölder\((1/2)\)*?
Theorem (Peres and S.)

Let $A$ be a closed set of $\mathbb{R}^d$, for $d \geq 2$, and $f$ a Hölder(1/2) continuous function. If $\mathbb{P}_x(B \text{ hits } A) > 0$, for all $x \in \mathbb{R}^d$, then $\mathbb{P}_x(B + f \text{ hits } A) > 0$, for all $x \in \mathbb{R}^d$. 

In 2 dimensions, if $\mathbb{P}_x(B \text{ hits } A) > 0$, then by neighborhood recurrence, $\mathbb{P}_x(B \text{ hits } A) = 1$. The same is true for $B + f$, if $f$ is Hölder(1/2).

Concerning the existence of multiple points, $B + f$ behaves in the same way as $B$, if $f$ is Hölder(1/2). (This can fail if $f$ is not Hölder(1/2), e.g. for $f$ fractional Brownian motion.)
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Hausdorff dimension

Definition (Hausdorff dimension)

For every $\alpha \geq 0$, the $\alpha$-Hausdorff content of a metric space $E$ is defined as:

$$H^\alpha_\infty(E) = \inf \left\{ \sum_{i=1}^\infty \left( \text{diam}(E_i) \right)^\alpha : E_1, E_2, \ldots \text{is a covering of } E \right\}.$$ 

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Let $B$ be a standard Brownian motion in $d \geq 1$ dimensions and let $f$ be a continuous function, $f : [0, 1] \rightarrow \mathbb{R}^d$. From our 0-1 law, we know that $\dim(B + f)[0, 1]$ is a constant a.s. Can we provide bounds for $\dim(B + f)[0, 1]$?
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Theorem (Peres and S.) 

$\dim(B_{[0,1]}) = 2^\wedge d$ a.s.

$\dim(B+f)_{[0,1]} \geq \max\{2^\wedge d, \dim(f_{[0,1]})\}$ a.s.
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Recall that $\dim B[0, 1] = 2 \wedge d$ a.s.
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$$\dim(B + f)[0, 1] \geq \max\{2 \wedge d, \dim f[0, 1]\} \quad a.s.$$
Let $B$ be a $d$ dimensional standard Brownian motion and let $f$ be a continuous function, $f : [0, 1] \rightarrow \mathbb{R}^d$. 

**Theorem (0-1 law for $L$)**

$\mathbb{P}(L(B+f)[0,1]) \in \{0, 1\}$. 

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**Theorem (0-1 law for $\mathcal{L}$)**

$$\mathbb{P}(\mathcal{L}(B + f)[0, 1] > 0) \in \{0, 1\}.$$
Proof of the 0-1 law for $\mathcal{L}$

For an interval $I \subset [0, 1]$, define $\Psi(I) = \mathcal{L}(B + f)(I)$. 

Declare $I \in D_n$ good if $\Psi(I) > 0$. Write $p_I = P(\Psi(I) > 0)$.

Let $Z_n$ be the number of good intervals of $D_n$. Then $Z_n$ is increasing in $n$.

Hence $E[Z_n] = \sum_{I \in D_n} p_I$ must be increasing.

The limit of $E[Z_n]$ exists and can be either infinite or finite.

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For an interval $I \subset [0, 1]$, define $\Psi(I) = \mathcal{L}(B + f)(I)$.

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The limit of \( \mathbb{E}[Z_n] \) exists and can be either infinite or finite.
Case 1: $\mathbb{E}[Z_n] = \sum_{I \in \mathcal{D}_n} p_I \uparrow \infty$
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Recall $\Psi(I) = \mathcal{L}(B + f)(I)$

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Letting $n \to \infty$ gives $\mathbb{P}(\Psi([0, 1]) = 0) = 0$. 

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Case 2: \( \mathbb{E}[Z_n] = \sum_{I \in D_n} p_I \uparrow C < \infty \)
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[0, 1] is the union of the good points and the dyadic intervals that do not contain any good points.

Since $\Psi(\text{good points}) = 0 \Rightarrow \Psi([0, 1]) = 0$ a.s.
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Declare $x \in [0, 1]$ good if all dyadic intervals that contain it are good.
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$I$ contains a good point $\iff \Psi(I) > 0$. 
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If $|\{\text{good points } \in [0, 1]\}| = \infty \Rightarrow Z_n \to \infty$, contradiction.
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Hence, $|\{\text{good points } \in [0,1]\}| < \infty$. 

[0,1] is the union of the good points and the dyadic intervals that do not contain any good points. Since $\Psi(\text{good points}) = 0 \Rightarrow \Psi([0,1]) = 0$ a.s.
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Proof of the 0-1 law for $\mathcal{L}$

**Case 2:** $\mathbb{E}[Z_n] = \sum_{I \in \mathcal{D}_n} p_I \uparrow C < \infty$

Recall $\Psi(I) = \mathcal{L}(B + f)(I)$

$Z_n = \sum_{I \in \mathcal{D}_n} 1(\Psi(I) > 0)$

Declare $x \in [0, 1]$ **good** if all dyadic intervals that contain it are good.

$I$ contains a good point $\iff \Psi(I) > 0$.

If $|\{\text{good points } \in [0, 1]\}| = \infty \Rightarrow Z_n \to \infty$, contradiction.

Hence, $|\{\text{good points } \in [0, 1]\}| < \infty$.

$[0, 1]$ is the union of the good points and the dyadic intervals that do not contain any good points.

Since $\Psi(\text{good points}) = 0 \Rightarrow \Psi([0, 1]) = 0$ a.s.
Let \((B_t, 0 \leq t \leq 1)\) be a standard Brownian motion in \(\mathbb{R}^d\), let \(f : [0, 1] \to \mathbb{R}^d\) be a continuous function and \(A\) a closed set in \([0, 1]\).
Let \((B_t, 0 \leq t \leq 1)\) be a standard Brownian motion in \(\mathbb{R}^d\), let \(f : [0, 1] \to \mathbb{R}^d\) be a continuous function and \(A\) a closed set in \([0, 1]\).

Which of the following events satisfy a 0-1 law?
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f: [0, 1] \rightarrow \mathbb{R}^d \] be a continuous function and \(A\) a closed set in [0, 1].

Which of the following events satisfy a 0-1 law?

- \(\{\mathcal{L}(B + f)(A) > 0\}\)
Let $(B_t, 0 \leq t \leq 1)$ be a standard Brownian motion in $\mathbb{R}^d$, let $f : [0, 1] \rightarrow \mathbb{R}^d$ be a continuous function and $A$ a closed set in $[0, 1]$. Which of the following events satisfy a 0-1 law?

- $\{ \mathcal{L}(B + f)(A) > 0 \}$
- $\{ \text{interior of } (B + f)(A) \neq \emptyset \}$
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- \( \{ \text{dim}(B + f)(A) > c \} \)
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- \(\{\mathcal{L}(B + f)(A) > 0\}\) \(\checkmark\)
- \(\{\text{interior of } (B + f)(A) \neq \emptyset\}\) \(\checkmark\)
- \(\{\text{dim}(B + f)(A) > c\}\)
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More on 0-1 laws

Let \((B_t, 0 \leq t \leq 1)\) be a standard Brownian motion in \(\mathbb{R}^d\), let \(f : [0, 1] \rightarrow \mathbb{R}^d\) be a continuous function and \(A\) a closed set in \([0, 1]\).

Which of the following events satisfy a 0-1 law?

- \(\{\mathcal{L}(B + f)(A) > 0\}\)  
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Which of the following events satisfy a 0-1 law?

- \(\{\mathcal{L}(B + f)(A) > 0\}\)  ✓
- \(\{\text{interior of } (B + f)(A) \neq \emptyset\}\)  ✓
- \(\{\text{dim}(B + f)(A) > c\}\)  ✓
- \(\{B \text{ is 1-1 on } A\}\)  ✗