Strong law of large numbers for the capacity of the Wiener sausage in dimension four

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Abstract

We prove a strong law of large numbers for the capacity of a Wiener sausage in dimension four: the capacity divided by its mean converges almost surely to one. We also obtain upper and lower bounds on the mean, and conjecture that our lower bound is sharp.
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1 Introduction

We denote by $(\beta_s, s \geq 0)$ a Brownian motion on $\mathbb{R}^4$, and for $r > 0$ and $0 \leq s \leq t \leq \infty$, let

$$W_r(s,t) = \{z \in \mathbb{R}^4 \ : \ |z - \beta_u| \leq r \ \text{ for some } s \leq u \leq t\}, \tag{1.1}$$

be the Wiener sausage of radius r in the time interval $[s,t]$. Let $\mathbb{P}_x$ and $\mathbb{E}_x$ be the law and expectation with respect to the Brownian motion started at site x, and let G denote Green's function and H_A denote the hitting time of A. The Newtonian capacity of a compact set $A \subset \mathbb{R}^4$ may be defined as

$$\mathrm{Cap}(A) = \lim_{|x|\to\infty} \frac{\mathbb{P}_x[H_A < +\infty]}{G(x)}. \tag{1.2}$$

Our main result is the following:

Theorem 1.1. *In dimension four, almost surely,*

$$\lim_{t\to\infty} \frac{\mathrm{Cap}(W_1(0,t))}{\mathbb{E}[\mathrm{Cap}(W_1(0,t))]} \ = \ 1. \tag{1.3}$$

Moreover, there exists a constant $C > 0$, such that for all $t \geq 2$,

$$(1 + o(1))\, \pi^2 \cdot \frac{t}{\log t} \ \leq \ \mathbb{E}[\mathrm{Cap}(W_1(0,t))] \ \leq \ C \cdot \frac{t}{\log t}. \tag{1.4}$$

We conjecture that the lower bound in (1.4) is sharp, based on results of Albeverio and Zhou, see Remark 3.2 below.

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From (1.2), the capacity of Wiener sausage is equivalent to the probability that two Wiener sausages intersect. Estimating such a probability has a long tradition: pioneering works were produced by Dvoretzky, Erdös and Kakutani [4] and Aizenman [1]; Aizenman's results have been subsequently improved by Albeverio and Zhou [2], Peres [14], Pemantle, Peres and Shapiro [13] and Khoshnevisan [7] (and references therein). In the discrete setting, the literature is even larger and older, and analogous results are presented in Lawler's comprehensive book [8].

We note that the problem of obtaining a law of large numbers for the capacity of the Wiener sausage has been raised recently by van den Berg, Bolthausen and den Hollander in connection with torsional rigidity [15] – a new geometrical characteristic of the Wiener sausage on a torus.

The proof of (1.3) presents some similarities with the proof in the discrete case, which is given in our companion paper [3], but also some substantial differences. One remarkable feature is that in spite of not having an estimate for $\mathbb{E}[\mathrm{Cap}(W_1(0,t))]$, we obtain an almost sure asymptotics only based on recent large deviation estimates of Erhard and Poisat [5]. In particular the upper bound in (1.4) directly follows from their results. The lower bound in (1.4) on the other hand is obtained via the Paley-Zygmund inequality and standard estimates on the volumes of one, or the intersection of two Wiener sausages.

It may seem odd that the fluctuations result we obtain in our analysis of the discrete model [3] are not directly transposable in the continuous setting. However, it has been noticed some thirty years ago by Le Gall [9] that *it does not seem easy to deduce Wiener sausage estimates from random walks estimates*, and vice-versa. Let us explain one reason for that. The capacity of set A can be represented as the integral of the *equilibrium measure* of set A, very much as in the discrete formula for the capacity of the range $\mathcal{R}[0,n]$ of a random walk (with obvious notation)

$$\mathrm{Cap}(\mathcal{R}[0,n]) = \sum_{x\in\mathcal{R}[0,n]} P_x(H^+_{\mathcal{R}[0,n]} = \infty).$$

Whereas Lawler [8] has established deep non-intersection results for two random walks in dimension four, the corresponding results for the equilibrium measure of $W_1(0,t)$ are still missing.

It is interesting to note that the decomposition formula that we use for the capacity of the union of two sets is of a different nature to the one presented in [3] for the discrete setting. For any two compact sets A and B and for any r larger than the inradius of A and B

$$\mathrm{Cap}(A\cup B) = \mathrm{Cap}(A) + \mathrm{Cap}(B) - \chi_r(A,B) - \varepsilon_r(A,B), \tag{1.5}$$

where we use the notation $\mathcal{S}_r$ for the boundary of the ball of radius r, and

$$\chi_r(A,B) = 2\pi^2\, r^2\cdot\frac{1}{|\mathcal{S}_r|}\int_{S_r}(\mathbb{P}_z[H_A<H_B<\infty] + \mathbb{P}_z[H_B<H_A<\infty])\,dz, \tag{1.6}$$

and

$$\varepsilon_r(A,B) = 2\pi^2 r^2\cdot\frac{1}{|\mathcal{S}_r|}\int_{S_r}\mathbb{P}_z[H_A=H_B<\infty]\,dz.$$

Observe in particular that $\varepsilon_r(A,B)\le\mathrm{Cap}(A\cap B)$. The right hand side of (2.5) only involves intersection (or equivalently hitting) probabilities. This makes some notable difference in the way we bound the cross term χ (see Subsection 2.2 below).

The paper is organised as follows. Section 2 contains preliminary results: in Section 2.1 we gather some well-known facts about Brownian motion; in Section 2.2 we give a decomposition formula for the capacity of the union of two sets; in Section 2.3 we recall some known results about capacity and some large deviations estimates for the capacity of a Wiener sausage, due to Erhard and Poisat. We then give a corollary concerning intersection probabilities of Wiener sausages. The proof of Theorem 1.1 is given in Section 3: we first prove (1.4) in Section 3.1. Then Section 3.2 deals with the second moment of the cross term χ appearing in the decomposition of the capacity mentioned above, and eventually we conclude the proof in Section 3.3.

Notation: We write $f \lesssim g$, for two functions f and g, when there exists a constant $C > 0$, such that $f(x) \le C \cdot g(x)$, for all x and $f \asymp g$ when $f \lesssim g$ and $g \lesssim f$.

2 Preliminaries

2.1 Further notation and basic estimates

We denote by $\mathbb{P}_z$ the law of a Brownian motion starting from z, and simply write $\mathbb{P}$ when z is the origin. Likewise $\mathbb{P}_{z,z'}$ will denote the law of two independent Brownian motions starting respectively from z and z', and similarly for $\mathbb{P}_{z,z',z''}$. For any $x \in \mathbb{R}^4$ and $r > 0$, we denote by $\mathcal{S}(x,r)$ and $\mathcal{B}(x,r)$ respectively the sphere and the ball of radius r centered at x, that we abbreviate in $\mathcal{S}_r$ and $\mathcal{B}_r$ when x is the origin. We write $|A|$ for the Lebesgue measure of a Borel set A. We recall, see Theorem 3.33 in [12], that

$$G(x) \;=\; \frac{1}{2\pi^2} \cdot \frac{1}{|x|^2}, \tag{2.1}$$

Recall also, see Corollary 3.19 in [12], that for any $z \in \mathbb{R}^4$, with $|z| > r$,

$$\mathbb{P}_z[H_{\mathcal{B}_r} < \infty] = \frac{r^2}{|z|^2}. \tag{2.2}$$

We will furthermore need the following well-known estimate (see Remark 2.22 in [12]): there exist positive constants c and C, such that for any $t > 0$ and $r > 0$

$$\mathbb{P}\Big[\sup_{s\le t} |\beta_s| > r\Big] \;\le\; C \cdot \exp(-c\, r^2/t). \tag{2.3}$$

2.2 A decomposition formula for the capacity of two sets

For $A \subset \mathbb{R}^4$ define

$$\mathrm{rad}(A) := \sup\{|z| \;:\; z \in A\},$$

the inradius of A. If A is a compact subset of $\mathbb{R}^4$, then for any $r \ge \mathrm{rad}(A)$, one has:

$$\begin{aligned} \mathrm{Cap}(A) \;&=\; \lim_{|x|\to\infty} \frac{\mathbb{P}_x[H_A < \infty]}{G(x)} = \lim_{|x|\to\infty} \frac{\mathbb{P}_x[H_{\mathcal{S}_r} < \infty]}{G(x)} \cdot \int_{\mathcal{S}_r} \mathbb{P}_z[H_A < \infty]\, d\rho_x(z) \\ &=\; 2\pi^2\, r^2 \cdot \frac{1}{|\mathcal{S}_r|} \int_{\mathcal{S}_r} \mathbb{P}_z[H_A < \infty]\, dz, \end{aligned} \tag{2.4}$$

where ρ_x is the law of the Brownian motion starting from x at time $H_{\mathcal{S}_r}$, conditioned on the event that this hitting time is finite. The second equality above follows from the Markov property, and the last equality just expresses the fact that the harmonic measure from infinity of a ball, which by Theorem 3.46 in [12] is also the weak limit of ρ_x as x goes to infinity, is the uniform measure on the boundary of the ball, by rotational invariance of the law of Brownian motion.

We deduce from (2.4) that for any two compact sets A and B, and for any r larger than the inradius of A and B,

$$\mathrm{Cap}(A \cup B) = \mathrm{Cap}(A) + \mathrm{Cap}(B) - \chi_r(A,B) - \varepsilon_r(A,B), \tag{2.5}$$

where

$$\chi_r(A,B) = 2\pi^2\, r^2 \cdot \frac{1}{|\mathcal{S}_r|} \int_{S_r} (\mathbb{P}_z[H_A < H_B < \infty] + \mathbb{P}_z[H_B < H_A < \infty])\, dz, \tag{2.6}$$

and

$$\varepsilon_r(A,B) = 2\pi^2 r^2 \cdot \frac{1}{|\mathcal{S}_r|} \int_{S_r} \mathbb{P}_z[H_A = H_B < \infty]\, dz.$$

Observe in particular that $\varepsilon_r(A,B) \leq \mathrm{Cap}(A \cap B)$.

2.3 Intersection of Wiener sausages

Our aim in this subsection is obtain some bounds on the probability of intersection of two Wiener sausages, or equivalently on the hitting probability of a Wiener sausage. Our first tool for this is the following basic result (see Corollary 8.12 and Theorem 8.27 in [12]):

Lemma 2.1. *Let A be a compact set in $\mathbb{R}^4$. Then for any $x \in \mathbb{R}^4 \setminus A$,*

$$\mathbb{P}_x[H_A < \infty] \;\leq\; \frac{1}{2\pi^2\, d(x,A)^2} \cdot \mathrm{Cap}(A),$$

where $d(x,A) := \inf\{|x-y|^2 \,:\, y \in A\}$.

The other estimate we need was proved by Erhard and Poisat, see Equation (5.55) in [5]:

Lemma 2.2 (Erhard–Poisat [5])**.** *There exist positive constants t_0, c and R_0, such that for all $t \geq t_0$ and all $R \geq R_0$,*

$$\mathbb{P}\Big[\mathrm{Cap}(W_1(0,t)) \;\geq\; R \cdot \frac{t}{\log t}\Big] \;\leq\; t^{-c\,R}.$$

We deduce from these results the following proposition, which complements earlier results of Albeverio and Zhou [2], Khoshnevisan [7] and Pemantle, Peres, and Shapiro [13]:

Proposition 2.3. *There exist positive constants C and t_0, such that for all $t > t_0$ and all $z \in \mathbb{R}^4$, with $|z| > \sqrt{\log t}$,*

$$\mathbb{P}_{0,z}\left[H_{W_1(0,t)} < \infty\right] \;\leq\; C \cdot \frac{1 \wedge (t/|z|^2)}{\log t}, \tag{2.7}$$

and all z, z' with $|z|, |z'| > (\log t)^2$,

$$\mathbb{P}_{0,z,z'}\Big[H_{W_1(0,t)} < \infty,\; \widetilde{H}_{W_1(0,t)} < \infty\Big] \;\leq\; C \cdot \frac{(1 \wedge t/|z|^2) \cdot (1 \wedge t/|z'|^2)}{(\log t)^2}, \tag{2.8}$$

where H and $\widetilde{H}$ are the hitting times associated to two independent Brownian motions, independent of β, starting respectively from z and z'.

Proof. We first prove (2.7). In view of (2.2), it only amounts to bounding the term

$$\mathbb{P}_{0,z}\left[H_{W_1(0,t)} < \infty,\, W_1(0,t) \cap \mathcal{B}(z, r(z,t)) = \varnothing\right],$$

with $r(z,t) := 2(|z| \wedge \sqrt{t})(\log t)^{-1/2}$. Using next Lemma 2.2, we see that it suffices to bound the term

$$\mathbb{P}_{0,z}\left[H_{W_1(0,t)} < \infty,\ d(z, W_1(0,t)) \geq r(z,t),\ \mathrm{Cap}(W_1(0,t)) \leq C\frac{t}{\log t}\right],$$

with C some appropriate constant. Now by first conditioning on $W_1(0,t)$, and then applying Lemma 2.1, we deduce that the latter is bounded, up to a constant factor, by

$$\mathbb{E}\left[\frac{\mathbf{1}\{d(z,W_1(0,t)) \geq r(z,t)\}}{d(z,W_1(0,t))^2}\right] \cdot \frac{t}{\log t}.$$

Furthermore, on the event $\{d(z,W_1(0,t)) \geq r(z,t)\}$, since $|z|$ is assumed to be larger than $\sqrt{\log t}$, we also have

$$d(z, W_1(0,t)) \asymp d(z, \beta[0,t]).$$

Therefore all that remains to be done is to show that

$$\mathbb{E}\left[\sup_{s\leq t} \frac{1}{|z-\beta_s|^2}\right] \lesssim \frac{1}{\max(t,|z|^2)}. \tag{2.9}$$

Since the function $x \mapsto |x|^{-2}$ is harmonic on $\mathbb{R}^4 \setminus \{0\}$, if we define for $s \geq 0$,

$$M_s := |z - \beta_s|^{-2},$$

and

$$H_z = \inf\{s \ :\ \beta_s = z\},$$

then we know that $(M_{s\wedge H_z}, s \geq 0)$ is a positive martingale (see [12, Theorem 7.18]). However, since the point z is a polar set in dimension two and larger [12, Corollary 2.26], H_z is almost surely infinite, so that actually $(M_s, s \geq 0)$ itself is a martingale. Therefore by Doob's maximal inequality ([12, Proposition 2.43]),

$$\mathbb{E}\left[\sup_{s\leq t} M_s^{3/2}\right] \lesssim \mathbb{E}[M_t^{3/2}] \lesssim \frac{1}{\max(t^{3/2}, |z|^3)},$$

where the last inequality follows from simple computations. Using next Jensen's inequality this proves (2.9).

Now we prove (2.8). Suppose that H and $\widetilde{H}$ are the hitting times associated to γ and $\widetilde{\gamma}$, two independent Brownian motions, independent of β (and as a consequence independent of $W_1(0,t)$ as well). Define W^γ and $W^{\widetilde{\gamma}}$ to be the Wiener sausages associated to γ and $\widetilde{\gamma}$, and let

$$\sigma := \inf\{s \ :\ W_{1/2}(0,s) \cap W^{\gamma}_{1/2}(0,\infty) \neq \varnothing\},$$

and

$$\widetilde{\sigma} := \inf\{s \ :\ W_{1/2}(0,s) \cap W^{\widetilde{\gamma}}_{1/2}(0,\infty) \neq \varnothing\}.$$

Note that

$$\mathbb{P}_{0,z,z'}\left[H_{W_1(0,t)} < \infty,\ \widetilde{H}_{W_1(0,t)} < \infty\right] = \mathbb{P}_{0,z,z'}[\sigma < \widetilde{\sigma} \leq t] + \mathbb{P}_{0,z,z'}[\widetilde{\sigma} < \sigma \leq t]. \tag{2.10}$$

By symmetry one can just bound the first term in the right-hand side of (2.10). Now conditionally on γ, σ is a stopping time for β. In particular, conditionally on σ and β_σ, $W_1(\sigma,t)$ is equal in law to $\beta_\sigma + W_1'(0,t-\sigma)$, with W' a Wiener sausage, independent of everything else. Therefore

$$\mathbb{P}_{0,z,z'}[\sigma < \widetilde{\sigma} \leq t] \quad \leq \quad \mathbb{E}_{0,z}\left[\mathbf{1}\{\sigma \leq t\}\,\mathbb{P}_{0,z,z'}[\sigma < \widetilde{\sigma} \leq t \mid \sigma, \gamma, \beta_\sigma]\right]$$

$$\begin{aligned} &\leq \mathbb{E}_{0,z}\Big[\mathbf{1}\{\sigma \leq t\}\, \mathbb{P}_{0,z'-\beta_\sigma}[H_{W_1'(0,t-\sigma)} < \infty \mid \sigma]\Big] \\ &\leq \mathbb{E}_{0,z}\Big[\mathbf{1}\{\sigma \leq t\}\, \mathbb{P}_{0,z'-\beta_\sigma}[H_{W_1'(0,t)} < \infty]\Big]. \end{aligned}$$

Now we split the last expectation in two parts corresponding to the two events $\{|z'-\beta_\sigma| \leq \sqrt{\log t}\}$ and its complement, which we denote by E_1 and E_2 respectively:

$$E_1 := \mathbb{E}_{0,z}\Big[\mathbf{1}\{\sigma \leq t,\, |z'-\beta_\sigma| \leq \sqrt{\log t}\}\, \mathbb{P}_{0,z'-\beta_\sigma}[H_{W_1'(0,t)} < \infty]\Big],$$

and

$$E_2 := \mathbb{E}_{0,z}\Big[\mathbf{1}\{\sigma \leq t,\, |z'-\beta_\sigma| > \sqrt{\log t}\}\, \mathbb{P}_{0,z'-\beta_\sigma}[H_{W_1'(0,t)} < \infty]\Big].$$

For the first part E_1, set $D = |z'-\beta_\sigma|$, and

$$H'_{W_1(0,t)} := \inf\{s > H_{\mathcal{B}(z',\sqrt{\log t})} \ :\ \gamma_s \in W_1(0,t)\},$$

with γ a Brownian motion starting from z. Then by using (2.2) and (2.7), we get with the Markov property

$$\begin{aligned} E_1 \ \leq\ \mathbb{P}_{0,z}[\sigma \leq t,\, D \leq \sqrt{\log t}] \ &\leq\ \mathbb{P}_{0,z}[H_{\mathcal{B}(z',\sqrt{\log t})} < H'_{W_1(0,t)} \leq t] \\ &\lesssim\ \frac{\log t}{1+|z-z'|^2} \cdot \frac{1 \wedge t/|z'|^2}{\log t}, \end{aligned} \tag{2.11}$$

since any point at distance smaller than $\sqrt{\log t}$ from z', is at least at distance $\sqrt{\log t}$ from the origin, by triangular inequality (recall that by hypothesis $|z'| > (\log t)^2$). Now if

$$|z-z'| \ >\ R(z,t) := \frac{\log t}{1 \wedge \sqrt{t}/|z|}, \tag{2.12}$$

we are done with this first term. Otherwise we distinguish three cases. First if $|z|$ or $|z'|$ is larger than $t^{3/5}$, we are done just by an application of (2.3). If $t^{3/5} \geq |z| \geq \sqrt{t}$ and $|z-z'| \leq R(z,t)$, then $2t^{3/5} \geq |z'| \geq \sqrt{t}/2$, and since β has to hit the ball $\mathcal{B}(z,\sqrt{\log t})$, we are done by a single application of (2.2). Finally if $(\log t)^2 \leq |z| \leq \sqrt{t}$, one has using the Markov property for γ when it first exit the ball $\mathcal{B}(z,3R(z,t))$,

$$\begin{aligned} \mathbb{P}_{0,z,z'}[\sigma < \widetilde{\sigma} \leq t] \ &\leq\ \mathbb{P}[W_1(0,t) \cap \mathcal{B}(z,3R(z,t)) \neq \varnothing] \\ &\qquad + \frac{1}{|\mathcal{S}(z,3R(z,t))|} \int_{\mathcal{S}(z,3R(z,t))} \mathbb{P}_{0,z'',z'}[\sigma < \widetilde{\sigma} \leq t]\, dz'' \\ &\lesssim\ \frac{1}{(\log t)^2} + \frac{1}{|\mathcal{S}(z,3R(z,t))|} \int_{\mathcal{S}(z,3R(z,t))} \mathbb{P}_{0,z'',z'}[\sigma < \widetilde{\sigma} \leq t]\, dz'', \end{aligned}$$

using (2.2), and we are done as well, since then one can use the same proof as above with the points $z'' \in \mathcal{S}(z,3R(z,t))$ instead of z, which are at distance larger than $R(z,t)$ from z'.

It remains to bound the other part E_2. Using (2.7) yields

$$\begin{aligned} E_2 \ &=\ \mathbb{E}_{0,z}\Big[\mathbf{1}\{\sigma \leq t\}\, \mathbb{P}_{z'-\beta_\sigma}[H_{W_1'(0,t)} < \infty]\, \mathbf{1}\{D > \sqrt{\log t}\}\Big] \\ &\lesssim\ \frac{1}{\log t} \cdot \mathbb{E}_{0,z}\Big[\mathbf{1}\{\sigma \leq t\} \cdot (1 \wedge \frac{t}{D^2})\Big] \\ &\lesssim\ \frac{1}{\log t} \cdot \mathbb{E}_{0,z}\big[\mathbf{1}\{\sigma \leq t\} \cdot \mathbf{1}\{D \leq |z'|/4\}\big] + \frac{1}{(\log t)^2} \cdot (1 \wedge \frac{t}{|z|^2})(1 \wedge \frac{t}{|z'|^2}). \end{aligned}$$

Now consider

$$\tau_{z,z'} := \begin{cases} \inf\{s \ : \ \beta_s \in \mathcal{B}(z', |z'|/4\} & \text{if } |z-z'| > |z'|/2 \\ \inf\{s \ : \ \beta_s \in \mathcal{B}(z', 3|z'|/4) & \text{if } |z-z'| \le |z'|/2. \end{cases}$$

Note that by construction $|z - \beta_{\tau_{z,z'}}| \ge \max(|z-z'|, |z'|)/4$, and that on the event $\{D < |z'|/4\}$, one has $\sigma > \tau_{z,z'}$. Therefore by conditioning first on $\tau_{z,z'}$ and the position of β at this time, and then by using (2.7), we obtain for some constant $c > 0$,

$$\begin{aligned} \mathbb{E}_{0,z}\left[\mathbf{1}\{\sigma \le t\} \cdot \mathbf{1}\{D \le |z'|/4\}\right] &\lesssim \frac{1 \wedge t/|z-z'|^2}{\log t} \cdot \mathbb{P}[\tau_{z,z'} \le t] \\ &\lesssim \frac{1 \wedge t/|z-z'|^2}{\log t} \cdot e^{-c|z'|^2/t} \\ &\lesssim \frac{1}{\log t} \cdot (1 \wedge t/|z|^2) \cdot (1 \wedge t/|z'|^2), \end{aligned}$$

using (2.3) at the second line. Putting all pieces together, this concludes the proof of the proposition. □

3 Proof of Theorem 1.1

3.1 A first moment estimate

In this section we prove (1.4) that we state as a proposition:

Proposition 3.1. *There exist positive constants t_0 and C, such that for all $t \ge t_0$,*

$$(1 + o(1))\, \pi^2 \cdot \frac{t}{\log t} \ \le \ \mathbb{E}[\mathrm{Cap}(W_1(0,t))] \ \le \ C \cdot \frac{t}{\log t}.$$

Proof. The upper bound follows from Lemma 2.2, so we concentrate on the lower bound now. We first give a rough bound on the second moment of $\mathrm{Cap}(W_1(0,t))$. For this we use that if $A \subset B$, then $\mathrm{Cap}(A) \le \mathrm{Cap}(B)$, so for any compact set A, $\mathrm{Cap}(A) \le 2\pi^2 \mathrm{rad}(A)^2$, as the capacity of a ball of radius r is equal to $2\pi^2 r^2$. It follows that

$$\mathbb{E}[\mathrm{Cap}(W_1(0,t))^2] \ \lesssim \ \mathbb{E}[\mathrm{rad}(W_1(0,t))^4] \ \lesssim \ \mathbb{E}\Big[\sup_{s \le t} |\beta_s|^4\Big] \ \lesssim \ t^2. \tag{3.1}$$

Now we fix $r = \sqrt{t} \cdot \log t$, and using (2.3) and Cauchy-Schwarz we arrive at

$$\mathbb{E}[\mathrm{Cap}(W_1(0,t))] = \mathbb{E}[\mathrm{Cap}(W_1(0,t))\, \mathbf{1}\{\mathrm{rad}(W_1(0,t)) \le r\}] + \mathcal{O}(1).$$

Then using (2.4) and (2.3) again, we obtain

$$\mathbb{E}[\mathrm{Cap}(W_1(0,t))] = 2\pi^2\, r^2 \cdot \frac{1}{|\mathcal{S}_r|} \int_{\mathcal{S}_r} \mathbb{P}_{0,z}[H_{W_1(0,t)} < \infty]\, dz + \mathcal{O}(1). \tag{3.2}$$

So it just amounts now to finding a lower bound for the hitting probabilities of $W_1(0,t)$ when starting at distance r. For this we define

$$Z_t = |W_{1/2}(0,t) \cap \widetilde{W}_{1/2}(0,\infty)|,$$

with $\widetilde{W}$ a Wiener sausage independent of W. Then for any $z \neq 0$,

$$\mathbb{P}_{0,z}[H_{W_1(0,t)} < \infty] \;=\; \mathbb{P}_{0,z}[Z_t \neq 0] \;\geq\; \frac{\mathbb{E}_{0,z}[Z_t]^2}{\mathbb{E}_{0,z}[Z_t^2]}, \tag{3.3}$$

where the last inequality is Paley-Zygmund's inequality. Now

$$\begin{aligned}
\mathbb{E}_{0,z}[Z_t] \;&=\; \int \mathbb{P}[x \in W_{1/2}(0,t)] \cdot \mathbb{P}_z[x \in \widetilde{W}_{1/2}(0,\infty)]\, dx \\
&=\; \int \mathbb{P}[H_{\mathcal{B}(x,1/2)} \leq t] \cdot \mathbb{P}_z[H_{\mathcal{B}(x,1/2)} < \infty]\, dx \\
&=\; \int \mathbb{P}[H_{\mathcal{B}(x,1/2)} \leq t] \cdot (1 \wedge \frac{1}{4|z-x|^2})\, dx.
\end{aligned}$$

Assume now that $|z| = r = \sqrt{t} \cdot \log t$, and let $r' = \sqrt{t} \cdot \log\log t$. Using (2.2) and (2.3), leads to

$$\begin{aligned}
\mathbb{E}_{0,z}[Z_t] \;&=\; \int_{\mathcal{B}(0,r')} \mathbb{P}[H_{\mathcal{B}(x,1/2)} \leq t] \cdot (1 \wedge \frac{1}{4|z-x|^2})\, dx + \mathcal{O}\left(\frac{1}{(\log t)^4}\right) \\
&=\; \frac{1 + \mathcal{O}\left(\frac{\log\log t}{\log t}\right)}{4|z|^2} \int_{\mathcal{B}(0,r')} \mathbb{P}[H_{\mathcal{B}(x,1/2)} \leq t]\, dx + \mathcal{O}\left(\frac{1}{(\log t)^4}\right) \\
&=\; \frac{1 + \mathcal{O}\left(\frac{\log\log t}{\log t}\right)}{4|z|^2} \mathbb{E}[|W_{1/2}(0,t)|] + \mathcal{O}\left(\frac{1}{(\log t)^4}\right) \\
&\sim\; \frac{\pi^2}{8} \cdot \frac{1}{(\log t)^2},
\end{aligned} \tag{3.4}$$

where for the last line we use the classical result of Kesten, Spitzer, and Whitman on the volume of the Wiener sausage, see e.g. [10] or [11] and references therein, which implies:

$$\lim_{t\to\infty} \frac{1}{t} \cdot \mathbb{E}[|W_{1/2}(0,t)|] = \mathrm{Cap}(\mathcal{B}_{1/2}) = \frac{\pi^2}{2}. \tag{3.5}$$

It remains to bound the second moment of Z_t. To this end we will need Getoor's result [6] on the volume of the intersection of two Wiener sausages, see also [10] or [11], which implies

$$\lim_{t\to\infty} \frac{1}{\log t} \cdot \mathbb{E}[|W_{1/2}(0,t) \cap \widetilde{W}_{1/2}(0,\infty)|] = \frac{\mathrm{Cap}(\mathcal{B}_{1/2})^2}{4\pi^2} = \frac{\pi^2}{16}. \tag{3.6}$$

To simplify notation write $\tau_x = H_{\mathcal{B}(x,1/2)}$. Note that for any $x \neq x'$, one has a.s. $\tau_x \neq \tau_{x'}$, since, even when $|x - x'| < 1$, for the event $\{\tau_x = \tau_{x'}\}$ to hold, the Brownian motion has to hit a two-dimensional sphere which is a polar set, by Kakutani's theorem (see [12, Theorem 8.20]). Observe also that the function $\mathbb{P}_x[\tau_{x'} < \infty]$ is symmetric in x and x'. Therefore,

$$\begin{aligned}
\mathbb{E}_{0,z}[Z_t^2] \;&=\; \int\int \mathbb{P}[\tau_x \leq t,\, \tau_{x'} \leq t] \cdot \mathbb{P}_z[\tau_x < \infty,\, \tau_{x'} < \infty]\, dx\, dx' \\
&=\; 2\int\int \mathbb{P}[\tau_x \leq t,\, \tau_{x'} \leq t] \cdot \mathbb{P}_z[\tau_x < \tau_{x'} < \infty]\, dx\, dx' \\
&\leq\; \frac{1}{2}\int \frac{dx}{|z-x|^2} \int \mathbb{P}[\tau_x \leq t,\, \tau_{x'} \leq t] \cdot \mathbb{P}_x[\tau_{x'} < \infty]\, dx' \\
&=\; \int \frac{dx}{|z-x|^2} \int \mathbb{P}[\tau_x < \tau_{x'} \leq t] \cdot \mathbb{P}_x[\tau_{x'} < \infty]\, dx'
\end{aligned}$$

$$\leq \int \frac{dx}{|z-x|^2}\,\mathbb{E}\left[\mathbf{1}\{\tau_x \leq t\}\cdot \int \mathbb{P}_x[\tau_{x'} \leq t-\tau_x \mid \tau_x]\cdot \mathbb{P}_x[\tau_{x'}<\infty]\,dx'\right]$$

$$= \int \mathbb{E}\Big[\mathbf{1}\{\tau_x \leq t\}\cdot \mathbb{E}[Z_{t-\tau_x}\mid \tau_x]\Big]\,\frac{dx}{|z-x|^2}. \tag{3.7}$$

Now assume that $|z| = r = \sqrt{t}\cdot \log t$. Note that (3.5) and (3.6) give respectively

$$\int \mathbb{P}[t - t/\log t \leq \tau_x \leq t]\,dx \leq \mathbb{E}[|W_{1/2}(0, t/\log t)]\ \lesssim\ \frac{t}{\log t},$$

and

$$\mathbb{E}[Z_{t-\tau_x}\mid \tau_x]\ \leq\ \mathbb{E}[Z_t]\ \sim\ \frac{\pi^2}{16}\log t.$$

Therefore by cutting the last integral in (3.7) in two pieces, one on the set $\{|x| > r'\}$, and the other one on $\{|x| < r'\}$, we arrive, similarly as for the first moment computation, to

$$\begin{aligned}
\mathbb{E}_{0,z}[Z_t^2]\ &\leq\ \frac{1+o(1)}{t\cdot(\log t)^2}\cdot \int \mathbb{E}\Big[\mathbf{1}\{\tau_x \leq t - t/\log t\}\cdot \mathbb{E}[Z_{t-\tau_x}\mid \tau_x]\Big]\,dx\\
&\leq\ \frac{\pi^2}{16}\cdot\frac{1+o(1)}{t\cdot(\log t)^2}\int \mathbb{E}\left[\mathbf{1}\{\tau_x \leq t - t/\log t\}\cdot \log(t-\tau_x)\right]\,dx\\
&\leq\ \frac{\pi^4}{32}\cdot\frac{1+o(1)}{\log t}.
\end{aligned} \tag{3.8}$$

Now (3.2), (3.3), (3.4) and (3.8) finish the proof of the lower bound. □

Remark 3.2. We believe that the lower bound is sharp in Proposition 3.1. Indeed Albeverio and Zhou [2] have shown that for any positve function (a_t) converging to infinity and such that $\log(1 + t/a_t)/\log t$ goes to 0,

$$\mathbb{P}[\widetilde{W}_{1/2}(0,\infty)\cap W_{1/2}(a_t, a_t+t) \neq \varnothing] \sim \frac{1}{2}\log(1+\frac{t}{a_t})\cdot(\log t)^{-1}.$$

Now if we could apply this result by replacing a_t with the stopping time $\tau_t := \inf\{s\ :\ |\beta_s| = \sqrt{t}\cdot\log t\}$, whose mean is equal to $t(\log t)^2$, then injecting this estimate in (3.2) would give a matching upper bound. The main problem with this argument however, is that $\tau_t/t(\log t)^2$ is not concentrated: it is equal in law to τ_1 by scaling.

3.2 A second moment estimate

Here we estimate the second moment of the crossing term χ from the decomposition (2.5).

Proposition 3.3. *Let β and $\widetilde{\beta}$ be two independent Brownian motions and let W and $\widetilde{W}$ be their corresponding Wiener sausages. Then with $r(t) = \sqrt{t}\cdot\log t$,*

$$\mathbb{E}\Big[\chi_{r(t)}(W_1(0,t), \widetilde{W}_1(0,t))^2\,\mathbf{1}\{\mathrm{rad}(W_1(0,t)) \leq r(t),\ \mathrm{rad}(\widetilde{W}_1(0,t)) \leq r(t)\}\Big] = \mathcal{O}\left(\frac{t^2}{(\log t)^4}\right).$$

Proof of Proposition 3.3. First note that for any nonpolar compact sets A and B and any r larger than the inradius of A and B, we can upper bound $\chi_r(A,B)^2$ by

$$\chi_r(A,B)^2 \lesssim \frac{r^4}{|\mathcal{S}_r|^2}\int_{\mathcal{S}_r\times\mathcal{S}_r}\Big(\mathbb{P}_{z,z'}[H_A < H_B < \infty,\ \widetilde{H}_A < \widetilde{H}_B < \infty]$$

$$+\quad \mathbb{P}_{z,z'}[H_B < H_A < \infty,\, \widetilde{H}_B < \widetilde{H}_A < \infty] + \mathbb{P}_{z,z'}[H_A < H_B < \infty,\, \widetilde{H}_B < \widetilde{H}_A < \infty]$$
$$+\quad \mathbb{P}_{z,z'}[H_B < H_A < \infty,\, \widetilde{H}_A < \widetilde{H}_B < \infty]\Big)\, dz\, dz', \tag{3.9}$$

where H and $\widetilde{H}$ refer to the hitting times of two independent Brownian motions γ and $\widetilde{\gamma}$ starting respectively from z and z'. By using (2.2), we obtain

$$\begin{aligned} &\mathbb{P}_{z,z'}[H_A < H_B < \infty,\, \widetilde{H}_A < \widetilde{H}_B < \infty] \\ =\ &\mathbb{P}_{z,z'}\Big[H_A < H_B < \infty,\, \widetilde{H}_A < \widetilde{H}_B < \infty,\, H_{\mathcal{B}_{\sqrt{t}\cdot(\log t)^{-3}}} = \infty,\, \widetilde{H}_{\mathcal{B}_{\sqrt{t}\cdot(\log t)^{-3}}} = \infty\Big] + \mathcal{O}\left(\frac{1}{(\log t)^8}\right), \end{aligned}$$

where to simplify notation, we took $A = W_1(0,t)$, $B = \widetilde{W}_1(0,t)$, and $r = r(t)$ in the above equation. Now to bound the probability on the right-hand side one can use the Markov property at times H_A and $\widetilde{H}_A$ for γ and $\widetilde{\gamma}$ respectively, and we deduce

$$\begin{aligned} \mathbb{P}_{z,z'}[H_A < H_B < \infty,\, \widetilde{H}_A < \widetilde{H}_B < \infty] &\lesssim \mathbb{P}_{z,z'}\Big[H_A < \infty,\, \widetilde{H}_A < \infty\Big] \cdot \frac{1}{(\log t)^2} + \frac{1}{(\log t)^8} \\ &\lesssim \frac{1}{(\log t)^8}, \end{aligned} \tag{3.10}$$

using (2.8) twice. By symmetry we get as well

$$\mathbb{P}_{z,z'}[H_B < H_A < \infty,\, \widetilde{H}_B < \widetilde{H}_A < \infty] = \mathcal{O}\left(\frac{1}{(\log t)^8}\right). \tag{3.11}$$

Now to bound the last two terms in (3.9), we can first condition on $A = W_1(0,t)$ and $B = \widetilde{W}_1(0,t)$, and then using the inequality $ab \le a^2 + b^2$, together with (3.10) and (3.11), this gives

$$\begin{aligned} \mathbb{P}_{z,z'}[H_A < H_B < \infty,\, \widetilde{H}_B < \widetilde{H}_A < \infty] \le\ &\mathbb{P}_{z,z}[H_A < H_B < \infty,\, \widetilde{H}_A < \widetilde{H}_B < \infty] \\ &+ \mathbb{P}_{z',z'}[H_B < H_A < \infty,\, \widetilde{H}_B < \widetilde{H}_A < \infty] \\ =\ &\mathcal{O}\left(\frac{1}{(\log t)^8}\right). \end{aligned} \tag{3.12}$$

By symmetry it also gives

$$\mathbb{P}_{z,z'}[H_{\widetilde{W}_1(0,t)} < H_{W_1(0,t)} < \infty,\, \widetilde{H}_{W_1(0,t)} < \widetilde{H}_{\widetilde{W}_1(0,t)} < \infty] = \mathcal{O}\left(\frac{1}{(\log t)^8}\right). \tag{3.13}$$

Then the proof follows from (3.9), (3.10), (3.11), (3.12), and (3.13). □

3.3 Law of large numbers

The final step of the proof of Theorem 1.1 is entirely similar to the discrete case [3]. The only notable difference is that in the discrete case, to bound the error term ε, we used that the capacity of any set is bounded by its size. In the continuous setting this is no longer true, but one has the following

Lemma 3.4. *Let $\widetilde{W}$ be an independent copy of W. One has a.s. for all $t > 0$,*

$$\mathrm{Cap}(W_1(0,t) \cap \widetilde{W}_1(0,t)) \ \le\ C \cdot |W_3(0,t) \cap \widetilde{W}_3(0,t)|,$$

with $C = \mathrm{Cap}(\mathcal{B}(0,3))/|\mathcal{B}(0,1)|$.

Proof. Let $(\mathcal{B}(x_i,1))_{i\le M}$ be a finite covering of $W_1(0,t)\cap \widetilde{W}_1(0,t)$ by balls of radius one whose centers are all assumed to belong to say $\beta[0,t]$, the trace of the Brownian motion driving $W_1(0,t)$. Then by removing one by one some balls if necessary, one can obtain a sequence of disjoint balls $(\mathcal{B}(x_i,1))_{i\le M'}$, such that the enlarged balls $(\mathcal{B}(x_i,3))_{i\le M'}$ cover the set $W_1(0,t)\cap\widetilde{W}_1(0,t)$, and such that all of them intersect $W_1(0,t)\cap \widetilde{W}_1(0,t)$. But since the centers (x_i) also belong to $\beta[0,t]$, all the balls $\mathcal{B}(x_i,1)$ belong to the enlarged intersection $W_3(0,t)\cap \widetilde{W}_3(0,t)$. So one has on one hand

$$\mathrm{Cap}(W_1(0,t)\cap \widetilde{W}_1(0,t)) \ \le\ M'\cdot \mathrm{Cap}(\mathcal{B}(0,3)),$$

since for any A and B one has $\mathrm{Cap}(A\cup B)\le \mathrm{Cap}(A)+\mathrm{Cap}(B)$, and on the other hand

$$|\cup_{i=1}^{M'}\mathcal{B}(x_i,1)| \ =\ M'|\mathcal{B}(0,1)| \ \le\ |W_3(0,t)\cap \widetilde{W}_3(0,t)|,$$

since by construction the balls $\mathcal{B}(x_i,1)$, $i\le M'$, are disjoint. This proves the lemma. □

One can now conclude the proof. For the convenience of the reader we recall the argument:

Proof of Theorem 1.1. Let L be such that $(\log n)^4\le 2^L\le 2(\log n)^4$, so that $L\asymp \log\log n$. Let $r=\sqrt{t}\cdot \log t$, and let E be the event $\{\mathrm{rad}(W_1(0,t))\le r\}$. On this event, by applying successively the decomposition formula (2.4) we obtain

$$\mathrm{Cap}(W_1(0,t)) = \sum_{i=1}^{2^L} X_{i,L}(t) + \sum_{\ell=1}^{L}\sum_{i=1}^{2^{\ell-1}} \chi_{i,\ell}(t) + \varepsilon(t),$$

where $\varepsilon(t)$ is a sum of error terms,

$$X_{i,L}(t) = \mathrm{Cap}\Big(W_1((i-1)2^{-L}t, i2^{-L}t)\Big),$$

and

$$\chi_{i,\ell}(t) = \chi_r\Big(W_1((2i-2)2^{-\ell}t,(2i-1)2^{-\ell}t), W_1((2i-1)2^{-\ell}t, i2^{-\ell+1}t)\Big),$$

with χ_r given by (2.6). Note that χ_r is defined independently of the fact that E holds or not. Thus the $(X_{i,L})_i$ are i.i.d., and for any fixed $\ell\le L$, the $(\chi_{i,\ell}(t))_i$ also.

Recall that by using (2.3), Proposition 3.3, we know that

$$\mathbb{E}[\chi_{i,\ell}(t)^2] \ =\ \mathcal{O}\left(\frac{t^2}{2^{2\ell}(\log t)^4}\right), \tag{3.14}$$

for all $i\le 2^{\ell-1}$. Therefore, letting

$$\chi_L(t) := \sum_{\ell=1}^{L}\sum_{i=1}^{2^{\ell-1}}\chi_{i,\ell}(t),$$

we get using the triangle inequality for the L^2-norm,

$$\mathrm{Var}\left(\chi_L(t)\right) \ \le\ L\cdot \sum_{\ell=1}^{L} 2^{\ell-1}\mathrm{Var}\left(\chi_{1,\ell}\right) \ =\ \mathcal{O}\left(t^2\cdot \frac{(\log\log t)^3}{(\log t)^4}\right).$$

Now fix some $\varepsilon>0$. Using Chebyshev's inequality we deduce from the previous bound that

$$\mathbb{P}\left[|\chi_L(t)-\mathbb{E}[\chi_L(t)]|\ge \varepsilon\frac{t}{\log t}\right] \ \lesssim\ \frac{(\log\log t)^3}{(\log t)^2}. \tag{3.15}$$

Next, using (3.1), and Chebyshev's inequality again gives

$$\mathbb{P}\left[\left|\sum_{i=1}^{2^L}(X_{i,L}(t)-\mathbb{E}[X_{i,L}(t)])\right|\geq \varepsilon\frac{t}{\log t}\right] \;=\; \mathcal{O}\left(\frac{1}{(\log t)^2}\right). \tag{3.16}$$

Then to finish the proof it remains to control the mean of $\varepsilon(t)$. By combining Lemma 3.4 with (3.6), we deduce that

$$\mathbb{E}\Big[\mathrm{Cap}(W_1(0,t)\cap \widetilde{W}_1(0,t))\Big] \;=\; \mathcal{O}(\log t).$$

So by definition of $\varepsilon(t)$ as a sum of the ε_r-terms appearing in Subsection 2.2, we obtain

$$\mathbb{E}[\varepsilon(t)] \;=\; \mathcal{O}(2^{L+1}\log t) \;=\; \mathcal{O}((\log t)^5).$$

A first consequence is that by Markov's inequality

$$\mathbb{P}\left[\varepsilon(t)\geq \varepsilon\frac{t}{\log t}\right] \;\lesssim\; \frac{(\log t)^6}{t}, \tag{3.17}$$

and another consequence is that

$$\mathbb{E}[\mathrm{Cap}(W_1(0,t)] \;\sim\; \sum_{i=1}^{2^L}\mathbb{E}[X_{i,L}(t)]+\mathbb{E}[\chi_L(t)]\,. \tag{3.18}$$

So putting all pieces together, namely (3.15), (3.16), (3.17) and (3.18), we arrive at the conclusion that

$$\mathbb{P}\left[|\mathrm{Cap}(W_1(0,t))-\mathbb{E}[\mathrm{Cap}(W_1(0,t))]\,|\geq 4\varepsilon\frac{t}{\log t}\right] \;=\; \mathcal{O}\left(\frac{(\log\log t)^3}{(\log t)^2}\right).$$

Now consider the sequence $a_n=\exp(n^{3/4})$. Since the previous bound holds for all $\varepsilon>0$, by using Borel-Cantelli's lemma and Proposition 3.1, we deduce that a.s.

$$\lim_{n\to\infty}\frac{\mathrm{Cap}(W_1(0,a_n))}{\mathbb{E}[\mathrm{Cap}(W_1(0,a_n))]} \;=\; 1.$$

Let now $t>0$, and choose $n=n(t)>0$, so that $a_n\leq t<a_{n+1}$. Using that the map $t\mapsto \mathrm{Cap}(W_1(0,t))$ is a.s. nondecreasing (since for any sets $A\subset B$, one has $\mathrm{Cap}(A)\leq\mathrm{Cap}(B)$), we can write

$$\frac{\mathrm{Cap}(W_1(0,a_n))}{\mathbb{E}[\mathrm{Cap}(W_1(0,a_{n+1}))]} \;\leq\; \frac{\mathrm{Cap}(W_1(0,t))}{\mathbb{E}[\mathrm{Cap}(W_1(0,t))]} \;\leq\; \frac{\mathrm{Cap}(W_1(0,a_{n+1}))}{\mathbb{E}[\mathrm{Cap}(W_1(0,a_n))]},$$

and the proof of the strong law of large numbers follows once we observe that

$$\mathbb{E}[\mathrm{Cap}(W_1(a_n,a_{n+1}))] \;\asymp\; \frac{a_{n+1}-a_n}{\log(a_{n+1}-a_n)} \;=\; o\left(\frac{a_n}{\log a_n}\right),$$

and use that for any sets A and B one has $\mathrm{Cap}(A)\leq\mathrm{Cap}(A\cup B)\leq\mathrm{Cap}(A)+\mathrm{Cap}(B)$. □

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