Symmetric Rearrangements Around Infinity with Applications to Lévy Processes

Alexander Drewitz [1] Perla Sousi [2] Rongfeng Sun [3]

October 25, 2011

Abstract

We prove a new rearrangement inequality for multiple integrals, which partly generalizes a result of Friedberg and Luttinger [FL76] and can be interpreted as involving symmetric rearrangements of domains around ∞. As applications, we prove two comparison results for general Lévy processes and their symmetric rearrangements. The first application concerns the survival probability of a point particle in a Poisson field of moving traps following independent Lévy motions. We show that the survival probability can only increase if the point particle does not move, and the traps and the Lévy motions are symmetrically rearranged. This essentially generalizes an isoperimetric inequality of Peres and Sousi [PS11] for the Wiener sausage. In the second application, we show that the q-capacity of a Borel measurable set for a Lévy process can only increase if the set and the Lévy process are symmetrically rearranged. This result generalizes an inequality obtained by Watanabe [W83] for symmetric Lévy processes.

AMS 2010 subject classification: Primary 26D15, 60J65. Secondary 60D05, 60G55, 60G50.

Keywords: capacity, isoperimetric inequality, Lévy process, Lévy sausage, Pascal principle, rearrangement inequality, trapping dynamics.

1 Introduction

1.1 Rearrangement Inequality

As motivation, let us start with the following random walk exit problem. Suppose that $(X_n)_{n\geq 0}$ is a discrete time random walk on $\mathbb{R}^d$ with transition probability kernel $p_n(x)\mathrm{d}x$ from time $n-1$ to n. Let $(A_n)_{n\geq 0}$ be a sequence of Borel-measurable sets in $\mathbb{R}^d$ with finite volume, such that the walk is killed at time i if $X_i \notin A_i$. If X_0 is uniformly distributed on A_0, then

$$\mathbb{P}(X_i \in A_i \ \forall\ 0 \leq i \leq n) = \frac{1}{|A_0|} \int \cdots \int \prod_{i=0}^{n} 1_{A_i}(x_i) \prod_{i=1}^{n} p_i(x_i - x_{i-1}) \prod_{i=0}^{n} \mathrm{d}x_i, \tag{1.1}$$

where $|A_0|$ denotes the Lebesgue measure of A_0. By the classic Brascamp-Lieb-Luttinger rearrangement inequality (see [BLL74] and [LL01, Theorem 3.8]), the above probability is upper bounded by

$$\frac{1}{|A_0^*|} \int \cdots \int \prod_{i=0}^{n} 1_{A_i^*}(x_i) \prod_{i=1}^{n} p_i^*(x_i - x_{i-1}) \prod_{i=0}^{n} \mathrm{d}x_i, \tag{1.2}$$

where A_i^* and p_i^* denote respectively the symmetric decreasing rearrangements of A_i and p_i, which are defined as follows.

[1] Departement Mathematik, ETH Zürich, Zürich, Switzerland. Email: alexander.drewitz@math.ethz.ch
[2] Statistical Laboratory, University of Cambridge, Cambridge, UK. Email: p.sousi@statslab.cam.ac.uk
[3] Department of Mathematics, National University of Singapore, Singapore. Email: matsr@nus.edu.sg

Definition 1.1 If $A \subset \mathbb{R}^d$ with $|A| < \infty$ (i.e., A has finite volume), then its symmetric decreasing rearrangement A^* is defined to be the open ball centered at the origin with $|A^*| = |A|$. If $|A| = \infty$, then we define $A^* := \mathbb{R}^d$. If $f : \mathbb{R}^d \to [0, \infty]$ is measurable, then its symmetric decreasing rearrangement f^* is defined to be

$$f^*(x) := \int_0^\infty 1_{F_t^*}(x)\, \mathrm{d}t, \qquad x \in \mathbb{R}^d,$$

where $F_t := \{y : f(y) > t\}$, $t \geq 0$, are the level sets of f (note that $f(x) = \int_0^\infty 1_{F_t}(x)\, \mathrm{d}t$). In particular, $f^*(x) = g(|x|)$ for a $g : [0, \infty) \to [0, \infty]$ which is nonincreasing and right-continuous.

In other words, in (1.1), the probability that the walk X survives up to time n can only increase if its transition kernels, as well as the domains, are all replaced by their symmetric decreasing rearrangements. There is a sizable literature on rearrangement inequalities and their relation to isoperimetric problems, see e.g. [LL01, Chapter 3]. Combined with probabilistic representations, rearrangement inequalities can be used to obtain the celebrated Rayleigh-Faber-Krahn inequality on the first eigenvalue of the Dirichlet Laplacian, and comparison inequalities for heat kernels and Green functions (see e.g. [BS01, BM-H10] and the references therein).

We are interested in the analogue of the survival probability in (1.1), where we replace the domains A_i by their complements A_i^c. Since $|A_i^c| = \infty$, the multiple integral in (1.1) is in general infinite if we replace A_i by A_i^c. However, it is sensible to consider instead

$$W_n((A_i)_{i\geq 0}, (p_i)_{i\geq 1}) := \int \cdots \int \Big(1 - \prod_{i=0}^n 1_{A_i^c}(x_i)\Big) \prod_{i=1}^n p_i(x_i - x_{i-1}) \prod_{i=0}^n \mathrm{d}x_i, \tag{1.3}$$

which can be interpreted as the total measure killed by the hard traps $(A_i)_{i\geq 0}$ by time n, if the initial measure of X_0 is the Lebesgue measure on $\mathbb{R}^d$ instead of a probability measure. The rearrangement inequality we will prove amounts to the statement that

$$W_n((A_i)_{i\geq 0}, (p_i)_{i\geq 1}) \geq W_n((A_i^*)_{i\geq 0}, (p_i^*)_{i\geq 1}). \tag{1.4}$$

Although (1.4) is still formulated in terms of symmetric decreasing rearrangements of A_i and p_i, with the origin being the center of rearrangements, it does not follow directly from classic rearrangement inequalities because terms with alternating signs appear when we expand $\prod_{i=0}^n (1 - 1_{A_i}(x_i))$. In both (1.1) and (1.3), the goal is to maximize the probability that the walk stays within the domains. The only difference is the replacement of the domains $(A_i)_{i\geq 1}$ in (1.1) by their complements in (1.3). In light of the close analogy between the two problems, it is instructive to think of (1.4) as a rearrangement inequality where the infinite domains A_i^c are symmetrically rearranged around ∞. This point of view will guide our proof.

We now formulate our rearrangement inequality for multiple integrals, which is a more general version of (1.4). We will assume that: The initial measure for X_0 is $\phi(x)\mathrm{d}x$ for some $\phi : \mathbb{R}^d \to [0, \infty)$; each hard trap A_i is replaced by a trap function $V_i : \mathbb{R}^d \to [0, 1]$, so that upon jumping to x_i at time i, the walk is killed with probability $V_i(x_i)$ instead of $1_{A_i}(x_i)$; each kernel $p_i : \mathbb{R}^d \to [0, \infty)$ is no longer assumed to be a probability density kernel.

Theorem 1.2 *Let $\phi : \mathbb{R}^d \to [0, \infty)$ and let $\sigma := \sup\{t \geq 0 : |\{x : \phi(x) < t\}| < \infty\}$. Define the symmetric increasing rearrangement of ϕ by $\phi_* := \sigma - (\sigma - \phi \wedge \sigma)^*$. For $i \geq 0$ and $j \geq 1$, let $V_i : \mathbb{R}^d \to [0, 1]$ and $p_j : \mathbb{R}^d \to [0, \infty)$, and let V_i^* and p_j^* denote their symmetric decreasing rearrangements. Denote $V_\cdot := (V_i)_{i\geq 0}$, $p_\cdot := (p_j)_{j\geq 1}$, $V_\cdot^* := (V_i^*)_{i\geq 0}$, and $p_\cdot^* := (p_j^*)_{j\geq 1}$. Then for all $n \geq 0$,*

$$\begin{aligned} W_n(\phi, V_\cdot, p_\cdot) &:= \int \cdots \int \phi(x_0)\Big(1 - \prod_{i=0}^n (1 - V_i(x_i))\Big) \prod_{i=1}^n p_i(x_i - x_{i-1}) \prod_{i=0}^n \mathrm{d}x_i \\ \geq W_n(\phi_*, V_\cdot^*, p_\cdot^*) &:= \int \cdots \int \phi_*(x_0)\Big(1 - \prod_{i=0}^n (1 - V_i^*(x_i))\Big) \prod_{i=1}^n p_i^*(x_i - x_{i-1}) \prod_{i=0}^n \mathrm{d}x_i. \end{aligned} \tag{1.5}$$

Remark 1.3 Theorem 1.2 partly generalizes an inequality of Friedberg and Luttinger [FL76], which is the special case of (1.5) in dimension $d = 1$, with $\phi \equiv 1$ and $p_i = p_i^*$ for all $i \geq 1$. They however allow an additional convolution kernel $p_{n+1}(x_0 - x_n)$ with $p_{n+1} = p_{n+1}^*$, which is set to 1 in our case. By the same reasoning as pointed out at the end of [FL76], if we include the additional kernel $p_{n+1}(x_0 - x_n)$, then the analogue of (1.5) is generally false. We will discuss two extensions of (1.5) in Remark 3.1.

It was pointed out by Méndez-Hernández in [M-H06, Theorem 2] that Friedberg and Luttinger's inequality also holds in higher dimensions. Recently, Peres and Sousi [PS11, Prop. 1.6] gave a new proof of this fact. More precisely, they proved (1.5) where $(V_i)_{i\geq 0}$ were taken to be indicator functions of open sets, and $(p_i)_{\geq 1}$ were taken to be the densities of uniform distributions on centered open balls. The interpretation of symmetric rearrangements around ∞ arises naturally in their proof. They appealed to an analogue rearrangement inequality on the sphere by Burchard and Schmuckenschläger [BS01, Theorem 2], which they applied by performing symmetric decreasing rearrangements of domains around the south pole of the sphere. As the radius of the sphere tends to infinity, the neighborhood around the north pole approximates $\mathbb{R}^d$, while the south pole converges to ∞. However, we are not aware of an analogue of (1.5) on the sphere which allows for symmetric decreasing rearrangements of the convolution kernels $(p_i)_{i\geq 1}$. Instead, we will develop a new approach to prove (1.5), which is based on induction and a proper notion of symmetric domination. This approach can also be used to prove rearrangement inequalities for multiple integrals of the type in (1.1).

Our primary motivation for Theorem 1.2 originates in the study of the survival probability of a point particle in a Poisson field of moving traps, each following an independent Lévy motion, which gives rise to continuous time analogues of the total killed measure W_n defined in (1.3). Therefore our first application of Theorem 1.2 is to show that the survival probability of the point particle can only increase if it stays put, while the Lévy motions and the shape of the traps are symmetric decreasingly rearranged (see Theorem 1.4). Previously, Peres and Sousi [PS11] proved such a comparison result when the traps follow independent Brownian motions, so that only the point particle motion and the shape of the traps require symmetric decreasing rearrangements. Our attempt to generalize their result to allow for symmetric decreasing rearrangements of general Lévy motions was inspired by the work of Bañuelos and Méndez-Hernández [BM-H10], where a continuous time analogue of the exit problem in (1.1) was considered. More specifically, they showed that the survival probability of a Lévy motion in a time-independent trap potential on a finite volume open domain can only increase if the Lévy motion and the domain are symmetric decreasingly rearranged, while the trap potential is symmetric increasingly rearranged[1].

Like classical rearrangement inequalities, Theorem 1.2 also has its potential-theoretic implications. As a second application of Theorem 1.2, we prove a comparison inequality for capacities of sets for Lévy processes (Theorem 1.9). More precisely, we show that if A is any Borel-measurable subset of $\mathbb{R}^d$, then the q-capacity of A for a Lévy process X ($q > 0$ if X is recurrent, and $q \geq 0$ if X is transient) can only decrease if we replace A and X by their symmetric decreasing rearrangements. This generalizes a result of Watanabe [W83], who proved such a comparison inequality for symmetric Lévy processes using Dirichlet forms. Special cases of Watanabe's result have been reproduced by Betsakos [B04] and Méndez-Hernández [M-H06]. An inequality of the type in Theorem 1.2 was in fact conjectured in [BM-H10], where its connection to 0-capacities was also pointed out.

In the remainder of this introduction, we will formulate precisely our comparison inequalities for the trap model and for capacities. We will then end the introduction with an outline of the rest of the paper.

[1] There was an error in the formulation of Theorem 1.4 in [BM-H10], where the symmetric decreasing rearrangement V^* of the potential V on the domain D should be replaced by its symmetric increasing rearrangement V_* on the domain D^*.

1.2 Trap Model

The model of a point particle in a Poisson field of moving traps in $\mathbb{R}^d$ is defined as follows. The point particle follows a deterministic path in $\mathbb{R}^d$, given by the function $f : [0,\infty) \to \mathbb{R}^d$. Let Ξ_0 be a Poisson point process on $\mathbb{R}^d$ with intensity measure $\phi(x)\mathrm{d}x$ for some $\phi : \mathbb{R}^d \to [0,\infty)$. We label the points in Ξ_0 by $(z_0^n)_{n\in\mathbb{N}}$. The points in Ξ_0 move independently in time, each following the law of a Lévy process $X := (X_t)_{t\geq 0}$. Namely, we replace Ξ_0 at time $t > 0$ by $\Xi_t := \{z_t^n : n \in \mathbb{N}\}$, where $z_t^n = z_0^n + X_t^n$, and $(X^n)_{n\in\mathbb{N}}$ are i.i.d. copies of X with $X_0 = 0$. The points in Ξ_t determine the location of traps at time t, and the actual shape of the traps at time t is determined by a trap potential $U_t : \mathbb{R}^d \to [0,\infty]$. More precisely, the field of traps at time t determine a potential

$$\mathcal{U}_t(x) := \sum_{n\in\mathbb{N}} U_t(x - z_t^n), \qquad x \in \mathbb{R}^d. \tag{1.6}$$

A point particle following the trajectory f is then killed with rate $\mathcal{U}_t(f(t))$ at time t, and the probability that the particle has survived the traps by time t is given by

$$\exp\Big\{ - \int_0^t \mathcal{U}_s(f(s))\,\mathrm{d}s \Big\}.$$

By absorbing f into the definition of U_t, i.e., replacing $U_t(\cdot)$ with $U_t(\cdot + f(t))$, we may assume without loss of generality that $f \equiv 0$, which we do from now on.

We are interested in upper bounds on the averaged survival probability

$$S_t := \mathbb{E}\Big[\exp\Big\{ - \int_0^t \mathcal{U}_s(0)\,\mathrm{d}s \Big\}\Big] = \mathbb{E}\Big[\exp\Big\{ - \sum_{n\in\mathbb{N}} \int_0^t U_s(-z_0^n - X_s^n)\,\mathrm{d}s \Big\}\Big], \tag{1.7}$$

where $\mathbb{E}$ denotes expectation w.r.t. Ξ_0 and $(X^n)_{n\in\mathbb{N}}$. By integrating out Ξ_0, we can rewrite (1.7) as

$$S_t = \exp\Big\{ - \int_{\mathbb{R}^d} w_t(x)\phi(x)\,\mathrm{d}x \Big\}, \tag{1.8}$$

where

$$1 - w_t(x) := v_t(x) := \mathbb{E}_0\Big[\exp\Big\{ - \int_0^t U_s(-x - X_s)\,\mathrm{d}s \Big\}\Big] = \mathbb{E}_x\Big[\exp\Big\{ - \int_0^t U_s(-X_s)\,\mathrm{d}s \Big\}\Big], \tag{1.9}$$

where $\mathbb{E}_x$ denotes expectation w.r.t. the Lévy process X with $X_0 = x$. We can interpret $w_t(x)$ as the probability that the Lévy process $-X$, with $X_0 = x$, is killed before time t by the trap $(U_t)_{t\geq 0}$. We will follow the convention that

$$\int_0^t U_s(-X_s)\,\mathrm{d}s := \infty \qquad \text{if } U_s(-X_s) = \infty \text{ for some } s \in [0,t), \tag{1.10}$$

so that the Lévy process $-X_s$ is killed when it hits the hard trap $D_s := \{x : U_s(x) = \infty\}$ for some $s < t$.

Analysis of the averaged survival probability S_t then becomes equivalent to the analysis of

$$W_t^X(\phi, U_\cdot) := \int_{\mathbb{R}^d} w_t(x)\phi(x)\,\mathrm{d}x, \tag{1.11}$$

which can be interpreted as the total measure of $-X$ killed by the trap $U_\cdot := (U_s)_{s\geq 0}$ up to time t, if X starts with initial measure $\phi(x)\mathrm{d}x$ on $\mathbb{R}^d$. Note that W_t^X is exactly the continuous time analogue of W_n in (1.5). In light of our discussion above, e^{-W_n} can also be interpreted as the averaged survival probability of a point particle in a Poisson field of moving traps in discrete time. As a corollary of Theorem 1.2, we will show that

$$W_t^X(\phi, U_\cdot) \geq W_t^{X^*}(\phi_*, U_\cdot^*),$$

where $U^*_\cdot := (U^*_s)_{s\geq 0}$, and X^* denotes the symmetric decreasing rearrangement of the Lévy process X, which we now define.

Recall that each Lévy process X with $X_0 = 0$ is uniquely characterized by a triple $(b, \mathbb{A}, \nu)$, called the characteristic of the Lévy process (see e.g. [B96, S99]), such that the characteristic function of X_t for any $t \geq 0$ is given by

$$\mathbb{E}_0[e^{i\langle \xi, X_t\rangle}] = e^{-t\Psi(\xi)},$$

where

$$\Psi(\xi) = -i\langle b, \xi\rangle + \frac{1}{2}\langle \mathbb{A}\xi, \xi\rangle + \int_{\mathbb{R}^d} (1 - e^{i\langle \xi, x\rangle} + i\langle \xi, x\rangle 1_{\{|x|<1\}})\nu(\mathrm{d}x). \tag{1.12}$$

Here $b \in \mathbb{R}^d$ is a deterministic drift, $\mathbb{A}$ is the $d \times d$ covariance matrix of the Brownian component of X, and ν is a measure on $\mathbb{R}^d$ with

$$\int_{\mathbb{R}^d} \frac{|x|^2}{1+|x|^2}\nu(\mathrm{d}x) < \infty \qquad \text{and} \qquad \nu(\{0\}) = 0.$$

The measure ν is called the Lévy measure of X and determines the jumps of X. When $b = 0$, $\mathbb{A} = 0$ and $\nu(\mathbb{R}^d) < \infty$, X is simply a compound Poisson process. Each Lévy process admits a version with càdlàg sample paths, i.e., paths that are right continuous with left hand limits, which we shall assume for X. If we denote by $\rho(x)\mathrm{d}x$ the absolutely continuous part of ν with respect to the Lebesgue measure, then the symmetric decreasing rearrangement of X is defined to be the Lévy process X^* with characteristic $(0, \mathbb{A}^*, \nu^*)$, where $\mathbb{A}^* := \mathrm{Det}(\mathbb{A})^{\frac{1}{d}}\mathbb{I}_d$ with $\mathbb{I}_d$ being the $d \times d$ identity matrix, and $\nu^*(\mathrm{d}x) = \rho^*(x)\mathrm{d}x$. This is the definition of X^* given in [W83, Section 2].

We are now ready to formulate our comparison result for the survival probability $S_t = e^{-W_t^X}$.

Theorem 1.4 *Let $\phi : \mathbb{R}^d \to [0, \infty)$, and let ϕ_* be its symmetric increasing rearrangement defined in Theorem 1.2. Let $U_\cdot(\cdot) : [0,\infty)\times\mathbb{R}^d \to [0,\infty]$ be measurable, and for each $s \geq 0$, $|\{x : U_s(x) > l\}| < \infty$ for some $l < \infty$. Assume that $D_s := \{x : U_s(x) = \infty\}$ are open sets satisfying the regularity condition*

(R) $\qquad \forall\, s \geq 0$ *and* $x \in D_s$, $\;\exists\, \delta > 0$, *s.t.* $y \in D_{s'}$ $\;\forall\, |y - x| < \delta$ *and* $s' \in [s, s+\delta)$.

Let X be a Lévy process with characteristic $(b, \mathbb{A}, \nu)$, and let X^* be its symmetric decreasing rearrangement. Let $W_t^X(\phi, U_\cdot)$ be defined from X, ϕ, and $(U_s)_{s\geq 0}$ as in (1.11), and let $W_t^{X^*}(\phi_*, U^*_\cdot)$ be defined analogously. Then for all $t \geq 0$,

$$W_t^X(\phi, U_\cdot) \geq W_t^{X^*}(\phi_*, U^*_\cdot). \tag{1.13}$$

Remark 1.5 Condition (R) guarantees that if the Lévy process $-X_s \in D_s$ for some $s \in [0, t)$, then $-X_{s'} \in D_{s'}$ for all $s' \in [s, s+\delta)$ for some $\delta > 0$, because $-X_s$ is almost surely right continuous in s. This ensures that our convention in (1.10) is a.s. consistent with the usual definition of integral. The assumption on the level sets of U_s will ensure that $U^*_\cdot$ also satisfies condition (R) (see the proof of (iii) of Claim 4.1). Some natural sufficient conditions for (R) include: $D_s = D$ is an open set independent of time; $D_s = D + g(s)$ for an open set D and a càdlàg path $g : [0,\infty) \to \mathbb{R}^d$; $\{(x, s) : s \geq 0, x \in D_s\}$ is an open set in $[0,\infty) \times \mathbb{R}^d$; D_s^c is right continuous in s with respect to the Hausdorff distance on the space of subsets of $\mathbb{R}^d$.

The trap model defined above and its lattice version have been studied extensively in the physics literature, where the motion of the point particle can also be random (see e.g. [BB02, MOBC04] and the references therein). It has also been studied as a detection problem in a mobile communication network (see e.g. [PSSS11] and the references therein). See also [CX11] for a recent study of the trap model with a renormalized Newtonian-type trap potential. A precursor to (1.13) in the literature is the special case when X is a Brownian motion, $\phi \equiv 1$, and $U_s(x) = \infty \cdot 1_{\{|x+f(s)|<1\}}$, where we recall that $f : [0,\infty) \to \mathbb{R}^d$ is the path of the point particle which was absorbed into the trap potential

$(U_s)_{s\geq 0}$. In this case, inequality (1.13) only rearranges the function f. More precisely, it asserts that the survival probability $e^{-W_t^X}$ is maximized if the point particle follows the constant function $f \equiv 0$. This type of result, where the optimal trajectory is the constant trajectory, has been called the *Pascal principle* in the physics literature. For the lattice version of the trap model, the Pascal principle was established in [MOBC04], see also [DGRS10, Corollary 2.1]. In the continuum setting above where the spherical hard traps follow independent Brownian motions, it was first established in dimension 1 in [PSSS11], assuming that f is continuous. Subsequently, Peres and Sousi [PS11] generalized it to higher dimensions and proved (1.13) for the case where X is a Brownian motion and $U_s = \infty \cdot 1_{D_s}$ for any open sets $(D_s)_{s\geq 0}$. Their work and the work of Bañuelos and Méndez-Hernández [BM-H10] inspired us to prove (1.13) in its current general form.

Since the result of Peres and Sousi in [PS11] was formulated as an isoperimetric inequality for the expected volume of a Wiener sausage, which does not resemble (1.13) in appearance, we recall here the connection. In (1.13), let $\phi \equiv 1$ and let $U_s(\cdot) := \infty \cdot 1_{D_s}(\cdot)$, where $D_s := D + g(s)$ for a finite volume open set $D \subset \mathbb{R}^d$ and a càdlàg $g : [0, \infty) \to \mathbb{R}^d$. Note that $U_\cdot(\cdot)$ satisfies the assumptions in Theorem 1.4. From (1.9), we obtain

$$w_t(x) = \mathbb{P}_x(-X_s \in D + g(s) \text{ for some } s \in [0, t)) = \mathbb{P}_0\Big(-x \in \bigcup_{s\in[0,t)} (D + X_s + g(s))\Big),$$

where $\mathbb{P}_x$ denotes expectation for the Lévy process X with $X_0 = x$. Then

$$W_t^X(1, U_\cdot) = \int_{\mathbb{R}^d} w_t(x)\, \mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}_0\big[1_{\{-x\in\bigcup_{s\in[0,t)}(D+X_s+g(s))\}}\big]\, \mathrm{d}x = \mathbb{E}_0\Big[\mathrm{Vol}\Big(\bigcup_{s\in[0,t)}(D + X_s + g(s))\Big)\Big],$$

where $\bigcup_{s\in[0,t)}(D + X_s + g(s))$ is the sausage generated by the Lévy process X with added drift g. Therefore in this case, (1.13) is equivalent to a comparison inequality for the expected volume of a Lévy sausage. We formulate this as a Corollary.

Corollary 1.6 *Let X be a Lévy process and let X^* be its symmetric decreasing rearrangement. Let $D \subset \mathbb{R}^d$ be a finite volume open set, and let $g : [0, \infty) \to \mathbb{R}^d$ be càdlàg. Then for all $t > 0$,*

$$\mathbb{E}_0\Big[\mathrm{Vol}\Big(\bigcup_{s\in[0,t)}(D + X_s + g(s))\Big)\Big] \geq \mathbb{E}_0\Big[\mathrm{Vol}\Big(\bigcup_{s\in[0,t)}(D^* + X_s^*)\Big)\Big]. \tag{1.14}$$

When X is a Brownian motion, (1.14) was proved in [PS11] with $D + g(s)$ replaced by any open set D_s, without even assuming the measurability of D_s in s. We will not attempt such generality here.

1.3 Comparison of Capacities

As a corollary of Theorem 1.4, we establish a comparison inequality for the capacities of Borel sets for Lévy processes. First we recall the definition of q-capacities for a Lévy process $X := (X_t)_{t\geq 0}$, with $q > 0$ when X is recurrent and $q \geq 0$ when X is transient.

For any $A \in \mathcal{B}(\mathbb{R}^d)$, the Borel σ-algebra on $\mathbb{R}^d$, let $T_A(X) := \inf\{t \geq 0 : X_t \in A\}$ denote the first hitting time of A by X. We will omit X from $T_A(X)$ when it is clear from the context with respect to which process T_A is being evaluated. Let $\mathbb{P}_x(\cdot)$ and $\mathbb{E}_x[\cdot]$ denote probability and expectation for X with $X_0 = x$, and let $\widehat{\mathbb{P}}_x(\cdot)$ and $\widehat{\mathbb{E}}_x[\cdot]$ denote the analogues for $\widehat{X} := -X$. We recall the following definition from [B96, p.49] for A either open or closed, and from [PS71, Def. 6.1] for general $A \in \mathcal{B}(\mathbb{R}^d)$.

Definition 1.7 (q-capacitary measure and q-capacities) *Let $q > 0$. For any $A \in \mathcal{B}(\mathbb{R}^d)$, the q-capacitary measure of A for the Lévy process X is defined to be*

$$\mu_A^q(B) := q \int_{\mathbb{R}^d} \mathbb{E}_x\big[e^{-qT_A} 1_{\{X_{T_A}\in B\}}\big]\, \mathrm{d}x \qquad \textit{for all } B \in \mathcal{B}(\mathbb{R}^d). \tag{1.15}$$

Its total mass $C_X^q(A) := \mu_A^q(\mathbb{R}^d) = q \int_{\mathbb{R}^d} \mathbb{E}_x[e^{-qT_A}]\, \mathrm{d}x$ is called the q-capacity of A.

Some basic properties of μ_A^q and C_X^q include:

- [PS71, Thm. 6.2] μ_A^q is the unique Radon measure supported on $\overline{A}$, the closure of A, with

$$(\mu_A^q G^q)(\mathrm{d}x) := \int_{\mathbb{R}^d} \mu_A^q(\mathrm{d}y) G^q(y, \mathrm{d}x) = \widehat{p}_A^q(x)\, \mathrm{d}x, \tag{1.16}$$

  where

$$G^q(y, B) := \int_0^\infty e^{-qt} \mathbb{P}_y(X_t \in B)\, \mathrm{d}t \qquad \text{for all } B \in \mathcal{B}(\mathbb{R}^d), \tag{1.17}$$

$$\widehat{p}_A^q(x) := \widehat{\mathbb{E}}_x[e^{-qT_A}]. \tag{1.18}$$

- [PS71, Prop. 6.4] $C_X^q(\cdot)$ is a Choquet capacity on $\mathcal{B}(\mathbb{R}^d)$. In particular, for all $A, B \in \mathcal{B}(\mathbb{R}^d)$,

$$C_X^q(A) \le C_X^q(B) \qquad \text{if } A \subset B, \tag{1.19}$$

$$C_X^q(A) = \inf\{C_X^q(O) : A \subset O,\, O \,\text{open}\} = \sup\{C_X^q(K) : K \subset A,\, K \,\text{compact}\}. \tag{1.20}$$

If X is transient, i.e., $\lim_{t\to\infty} |X_t| = \infty$ a.s., then one can also define its 0-capacity. We recall the following definition from [B96, Cor. 8, p.52] and [PS71, Prop. 8.3].

Definition 1.8 (0-capacitary measure and 0-capacities) *Suppose that X is transient. Let $A \in \mathcal{B}(\mathbb{R}^d)$ be relatively compact. Then μ_A^q converges weakly to a measure μ_A^0, which is called the 0-capacitary measure of A for X. Its total mass $C_X^0(A) := \mu_A^0(\mathbb{R}^d)$ is called the 0-capacity (or just capacity) of A. For general $A \in \mathcal{B}(\mathbb{R}^d)$, we define $C_X^0(A) := \sup\{C_X^0(K) : K \subset A, K \,\text{relatively compact}\}$.*

For relatively compact $A \in \mathcal{B}(\mathbb{R}^d)$, the analogues of (1.16) and (1.19)–(1.20) also hold [PS71, Prop. 8.2 & 8.4], provided we replace G^q by

$$G^0(y, B) := \int_0^\infty \mathbb{P}_y(X_t \in B)\, \mathrm{d}t \qquad \text{for all } B \in \mathcal{B}(\mathbb{R}^d), \tag{1.21}$$

and replace $\widehat{p}_A^q(x)$ by

$$\widehat{p}_A^0(x) := \widehat{\mathbb{P}}_x(T_A < \infty). \tag{1.22}$$

We can now state our comparison inequality for capacities.

Theorem 1.9 *Let X be a Lévy process with characteristic $(b, \mathbb{A}, \nu)$, and let X^* be its symmetric decreasing rearrangement. Then for any $q > 0$ ($q \ge 0$ if X is transient), and for any $A \in \mathcal{B}(\mathbb{R}^d)$, we have*

$$C_X^q(A) \ge C_{X^*}^q(A^*). \tag{1.23}$$

Remark 1.10 Theorem 1.9 was conjectured in [BM-H10, p.4050]. It extends a result of Watanabe [W83, Theorem 1], where (1.23) was proved for symmetric Lévy processes, i.e., X is equally distributed with $-X$ if $X_0 = 0$. For Riesz capacities, which correspond to radially symmetric α-stable processes, Watanabe's result has been reproduced by Betsakos in [B04]. For isotropic unimodal Lévy processes, Watanabe's result has been reproduced by Méndez-Hernández in [M-H06], which uses the Friedberg-Luttinger inequality discussed in Remark 1.3.

In [W83], Watanabe used the definition of q-capacities from the theory of Dirichlet forms for symmetric Markov processes. It is known that such a definition is equivalent to the probabilistic definition given here if X is a symmetric Lévy process. However, a precise reference seems hard to locate. Therefore we will sketch briefly why the two definitions are equivalent.

For a symmetric Lévy process X with characteristic $(0, \mathbb{A}, \nu)$, one can define a family of Dirichlet forms $\mathcal{E}_q(\cdot,\cdot)$ ($q > 0$ if X is recurrent and $q \geq 0$ if X is transient). If L denotes the generator of X, and $L_q u := Lu - qu$ for $u \in \mathcal{D}(L) \subset L^2(\mathbb{R}^d)$, then the domain of $\mathcal{E}_q$ equals $\mathcal{D}(\mathcal{E}_q) = \mathcal{D}(\sqrt{-L_q}) \subset L^2(\mathbb{R}^d)$, and

$$\mathcal{E}_q(u,v) = \int_{\mathbb{R}^d} (\sqrt{-L_q}u)(x)(\sqrt{-L_q}v)(x)\,\mathrm{d}x \qquad \text{for } u, v \in \mathcal{D}(\mathcal{E}_q),$$
$$\mathcal{E}_q(u,v) = -\int_{\mathbb{R}^d} u(x)(L_q v)(x)\,\mathrm{d}x \qquad \text{for } u \in \mathcal{D}(\mathcal{E}_q), v \in \mathcal{D}(L_q);$$

see Theorem 1.3.1 and Corollary 1.3.1 in [FOT11]. For any open set O, its q-capacity is defined by (see e.g. [W83] or [FOT11, Chap. 2])

$$C_X^q(O) := \inf\{\mathcal{E}_q(u,u) : u \in \mathcal{D}(\mathcal{E}_q), u \geq 1 \text{ a.e. on } O\}, \tag{1.24}$$

with $C_X^q(O) := \infty$ if the infimum is taken over an empty set. The q-capacity of a general Borel set $A \subset \mathbb{R}^d$ is defined to be

$$C_X^q(A) := \inf\{C_X^q(O) : A \subset O, O \text{ open}\}. \tag{1.25}$$

For any bounded open set O, Lemma 2.1.1 (and the remark before Lemma 2.1.8) in [FOT11] show that the infimum in (1.24) is achieved at a function $e_O^q \in \mathcal{D}(\mathcal{E}_q)$, called the q-equilibrium potential of O. Furthermore,

$$C_X^q(O) = \mathcal{E}_q(e_O^q, v) = \int_{\mathbb{R}^d} e_O^q(x)(-L_q v)(x)\,\mathrm{d}x \qquad \forall\, v \in \mathcal{D}(L_q) \text{ with } v = 1 \text{ a.e. on } O. \tag{1.26}$$

Lemma 4.2.1 and Theorem 4.3.3 in [FOT11] identify e_O^q with $p_O^q = \widehat{p}_O^q$ (since the Lévy process is symmetric) defined in (1.18) and (1.22). Therefore, choosing any $v \in C_c^\infty(\mathbb{R}^d) \subset \mathcal{D}(L_q)$ with $v = 1$ on $\overline{O}$, we obtain

$$C_X^q(O) = \int p_O^q(x)(-L_q v)(x)\,\mathrm{d}x = \int \mu_O^q(\mathrm{d}y) \int G^q(y, \mathrm{d}x)(-L_q v)(x) = \int \mu_O^q(\mathrm{d}y) v(y), \tag{1.27}$$

where we used (1.16), and the fact that $G^q(x, \mathrm{d}y)$ is the Green's kernel for the Lévy process X killed with rate q, which is a transient process with generator L_q, and hence $G^q(x, \mathrm{d}y)$ defines an integral operator which is the inverse of $-L_q$. Since $v = 1$ on $\overline{O}$ and μ_O^q is supported on $\overline{O}$, $C_X^q(O)$ in (1.27) coincides with our definition of $C_X^q(O)$ in Definitions 1.7–1.8. Since $C_X^q(\cdot)$ defined via (1.24)–(1.25) is also a Choquet capacity by [FOT11, Theorem 2.1.1] and hence satisfies (1.20), the coincidence of the two definitions of capacities extends from bounded open sets to all Borel-measurable sets.

1.4 Outline

The rest of the paper is organized as follows. In Section 2, we collect some basic properties of symmetric rearrangements. Theorems 1.2, 1.4 and 1.9 will then be proved respectively in Sections 3, 4 and 5. Some concluding remarks are in Section 6.

2 Properties of Symmetric Rearrangements

We collect here some basic properties of symmetric decreasing rearrangements. The reader may skip this section until the lemmas stated here are invoked. Lemmas 2.1 and 2.2 will be used to carry out approximations. Lemma 2.4 considers the symmetric decreasing rearrangement of the convolution of two functions. Lemma 2.5 considers the spatial symmetric decreasing rearrangement of a function which is continuous in space and time.

Lemma 2.1 *Let $\phi, \phi_n : \mathbb{R}^d \to [0, \infty]$, $n \in \mathbb{N}$, be such that $\phi_n(x) \uparrow \phi(x)$ as $n \to \infty$ for Lebesgue almost every $x \in \mathbb{R}^d$. Then $\phi_n^*(x) \uparrow \phi^*(x)$ for every $x \in \mathbb{R}^d$.*

Proof. Since $(\phi_n^*)_{n\in\mathbb{N}}$ and ϕ^* remain unchanged if $(\phi_n)_{n\in\mathbb{N}}$ and ϕ are modified on a set of Lebesgue measure 0, we may assume without loss of generality that $\phi_n(x) \uparrow \phi(x)$ for all $x \in \mathbb{R}^d$. We now define for all $t > 0$ the level sets

$$\Phi_n(t) := \{z : \phi_n(z) > t\} \quad \text{and} \quad \Phi(t) := \{z : \phi(z) > t\}. \tag{2.1}$$

Then by the assumption that $\phi_n \uparrow \phi$, we get that

$$\Phi_n(t) \uparrow \Phi(t) \text{ as } n \to \infty,$$

which implies that

$$\Phi_n^*(t) \uparrow \Phi^*(t) \text{ as } n \to \infty.$$

(Note that if $|A| = \infty$, then we define $A^* := \mathbb{R}^d$.) Therefore by the Monotone Convergence theorem,

$$\phi_n^*(x) = \int_0^\infty 1_{\Phi_n^*(t)}(x)\,\mathrm{d}t \uparrow \int_0^\infty 1_{\Phi^*(t)}(x)\,\mathrm{d}t \qquad \text{as } n \to \infty.$$

Since $\phi^*(x) = \int_0^\infty 1_{\Phi^*(t)}(x)\,\mathrm{d}t$, we obtain

$$\lim_{n\to\infty} \phi_n^*(x) = \phi^*(x).$$

Note that this convergence holds for every $x \in \mathbb{R}^d$. ∎

Lemma 2.2 *Let $\phi, \phi_n : \mathbb{R}^d \to [0,\infty)$, $n \in \mathbb{N}$, be uniformly bounded with uniformly bounded support, such that $\phi_n(x) \to \phi(x)$ as $n \to \infty$ for Lebesgue almost every $x \in \mathbb{R}^d$. Then $\phi_n^*(x) \to \phi^*(x)$ for Lebesgue almost every $x \in \mathbb{R}^d$.*

Remark 2.3 Lemma 2.2 is a correction of [BM-H10, Lemma 4.2], where $(\phi_n)_{n\in\mathbb{N}}$ were not assumed to have uniformly bounded support, and the conclusion can be seen to be false. Indeed, fix any $0 \neq v \in \mathbb{R}^d$. Then $\phi_n(x) := 1_{\{|x-nv|<1\}}$ converges pointwise to $\phi \equiv 0$, and yet $\phi_n^*(x) = 1_{\{|x|<1\}} \not\to \phi^* \equiv 0$.

Proof. As in the proof of Lemma 2.1, we may assume without loss of generality that $\phi_n(x) \to \phi(x)$ for every $x \in \mathbb{R}^d$. We will first show that $\liminf_{n\to\infty} \phi_n^*(x) = \phi^*(x)$ for Lebesgue a.e. x.

Since $(\phi_n)_{n\in\mathbb{N}}$ and ϕ are uniformly bounded with uniformly bounded support, by the Dominated Convergence theorem, we have

$$\int_{\mathbb{R}^d} \phi_n(x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}x \qquad \text{as } n \to \infty.$$

Since for any nonnegative f, we have $\int f(x)\,\mathrm{d}x = \int f^*(x)\,\mathrm{d}x$, we obtain

$$\int_{\mathbb{R}^d} \phi_n^*(x)\,\mathrm{d}x \to \int_{\mathbb{R}^d} \phi^*(x)\,\mathrm{d}x \qquad \text{as } n \to \infty. \tag{2.2}$$

For $t > 0$, let the level sets $(\Phi_n(t))_{n\in\mathbb{N}}$ and $\Phi(t)$ be defined as in (2.1), which have finite volume by the assumption that $(\phi_n)_{n\in\mathbb{N}}$ and ϕ have uniformly bounded support.

By the convergence of ϕ_n to ϕ, we have

$$\Phi(t) \subset \liminf_{n\to\infty} \Phi_n(t) = \bigcup_{k=1}^\infty \bigcap_{n\geq k} \Phi_n(t).$$

Using this, Lemma 2.1 applied to the corresponding indicator functions yields the second equality in

$$\Phi^*(t) \subseteq \Big(\bigcup_{k=1}^{\infty}\bigcap_{n\geq k}\Phi_n(t)\Big)^* = \bigcup_{k=1}^{\infty}\Big(\bigcap_{n\geq k}\Phi_n(t)\Big)^* \subseteq \bigcup_{k=1}^{\infty}\bigcap_{n\geq k}\Phi_n^*(t) = \liminf_{n\to\infty}\Phi_n^*(t).$$

We can now write

$$\begin{aligned}\phi^*(x) &= \int_0^\infty 1_{\{x\in\Phi^*(t)\}}\,\mathrm{d}t \leq \int_0^\infty 1_{\liminf \Phi_n^*(t)}(x)\,\mathrm{d}t \\ &\leq \liminf_{n\to\infty}\int_0^\infty 1_{\Phi_n^*(t)}(x)\,\mathrm{d}t = \liminf_{n\to\infty}\phi_n^*(x),\end{aligned}$$

where in the second inequality we used Fatou's Lemma. We thus showed $\phi^*(x) \leq \liminf_{n\to\infty}\phi_n^*(x)$ for all x. If we now integrate over all $x\in\mathbb{R}^d$ and use Fatou's lemma again, we obtain

$$\int_{\mathbb{R}^d}\phi^*(x)\,\mathrm{d}x \leq \int_{\mathbb{R}^d}\liminf_{n\to\infty}\phi_n^*(x)\,\mathrm{d}x \leq \liminf_{n\to\infty}\int_{\mathbb{R}^d}\phi_n^*(x)\,\mathrm{d}x = \int_{\mathbb{R}^d}\phi^*(x)\,\mathrm{d}x,$$

where the equality follows from (2.2). Therefore, we deduce that

$$\liminf_{n\to\infty}\phi_n^*(x) = \phi^*(x), \quad \text{for Lebesgue a.e. } x. \tag{2.3}$$

We will now finish the proof by showing that $\limsup_{n\to\infty}\phi_n^*(x) \leq \phi^*(x)$ for Lebesgue a.e. x. We define a new sequence of functions $f_n := \sup_{k\geq n}\phi_k(x)$. By the assumptions on $(\phi_n)_{n\in\mathbb{N}}$, $(f_n)_{n\in\mathbb{N}}$ are also uniformly bounded with uniformly bounded support. Clearly, $f_n \downarrow \phi$ as $n\to\infty$. Therefore, we may apply (2.3) to f_n instead of ϕ_n, and deduce that

$$\limsup_{n\to\infty} f_n^*(x) = \liminf_{n\to\infty} f_n^*(x) = \phi^*(x) \quad \text{for Lebesgue a.e. } x,$$

and where to obtain the first equality we used that $f_n^*(x)$ is a nonincreasing sequence and thus $\limsup_{n\to\infty} f_n^*(x) = \liminf_{n\to\infty} f_n^*(x)$. Since $\phi_n^*(x) \leq f_n^*(x)$ for all x, we obtain

$$\limsup_{n\to\infty}\phi_n^*(x) \leq \phi^*(x) \quad \text{for Lebesgue a.e. } x,$$

which concludes the proof. ∎

Lemma 2.4 *Suppose that $f,g:\mathbb{R}^d\to[0,\infty)$ and $f=f^*$, $g=g^*$. Then $f*g=(f*g)^*$.*

Proof. Since f and g are radially symmetric, so must be $f*g$. Since f and g are lower semi-continuous, for any $x_n\to x$, we have

$$(f*g)(x) = \int f(x-y)g(y)\,\mathrm{d}y \leq \int \liminf_{n\to\infty} f(x_n-y)g(y)\,\mathrm{d}y \leq \liminf_{n\to\infty}\int f(x_n-y)g(y)\,\mathrm{d}y = \liminf_{n\to\infty}(f*g)(x_n).$$

Therefore $f*g$ is also lower semi-continuous. It only remains to show that $f*g$ is radially nonincreasing. By writing

$$(f*g)(x) = \int_{\mathbb{R}^d}\int_0^\infty 1_{\{f(x-y)>s\}}\,\mathrm{d}s\int_0^\infty 1_{\{g(y)>t\}}\,\mathrm{d}t\,\mathrm{d}y = \int_0^\infty\int_0^\infty (1_{F_s}*1_{G_t})(x)\,\mathrm{d}s\,\mathrm{d}t,$$

where $F_s := \{x : f(x)>s\}$ and $G_t := \{x : g(x)>t\}$ are centered open balls, we only need to show that $(1_{F_s}*1_{G_t})(x)$ is radially nonincreasing. This is equivalent to showing that $|F_s\cap(G_t+\lambda x)|$ is nonincreasing in $\lambda\geq 0$ for any $x\neq 0$, which is a consequence of the Brunn-Minkowski inequality, see e.g. [A55, Theorem 1]. ∎

Lemma 2.5 *Let $U_s(x) : [0,\infty) \times \mathbb{R}^d \to [0,\infty)$ be continuous with compact support. For each $s \geq 0$, let $U_s^*(\cdot)$ denote the symmetric decreasing rearrangement of $U_s(\cdot)$. Then $U_\cdot^*(\cdot)$ is also continuous on $[0,\infty) \times \mathbb{R}^d$ with compact support.*

Proof. Clearly $U_\cdot^*(\cdot)$ has compact support. We first show that for each $s \geq 0$, U_s^* is continuous. By definition of U_s^*, there exists $f_s : [0,\infty) \to [0,\infty)$, which is non-increasing and right-continuous, such that $U_s^*(x) = f_s(|x|)$ for all $x \in \mathbb{R}^d$. If U_s^* is discontinuous at some $x_0 \in \mathbb{R}^d$, then f_s has a jump discontinuity at $|x_0|$, and we must have $|x_0| > 0$. In particular, we must have

$$0 < |\{x \in \mathbb{R}^d : U_s(x) > f_s(|x_0|)\}| = |\{x \in \mathbb{R}^d : U_s(x) > f_s(|x_0|) + \epsilon\}| < \infty \qquad \text{for some } \epsilon > 0.$$

However the equality cannot hold because U_s is continuous. Therefore f_s must be continuous, and hence U_s^* must be continuous as well.

Next we show that $f_s(r)$ is jointly continuous in $s \geq 0$ and $r \geq 0$. The continuity of $U_\cdot(\cdot)$ and Lemma 2.2 imply that for each $s \geq 0$ and for Lebesgue a.e. $x \in \mathbb{R}^d$, $U_t^*(x) \to U_s^*(x)$ as $t \to s$. This in turn implies that for Lebesgue a.e. $r \geq 0$, $f_t(r) \to f_s(r)$ as $t \to s$. Since $(f_s)_{s\geq 0}$ are all continuous, monotone, with uniformly bounded support, $f_t(\cdot)$ must converge uniformly to $f_s(\cdot)$ as $t \to s$. This establishes the joint continuity of $f_s(r)$ in $s, r \geq 0$, and hence $U_s^*(x)$ must also be jointly continuous in $s \geq 0$ and $x \in \mathbb{R}^d$. ∎

3 Proof of Theorem 1.2

Proof of Theorem 1.2. By replacing ϕ with $\phi \wedge \sigma$, we may assume without loss of generality that $\sigma = 1$, $\phi \in [0,1]$, and $|\{x : \phi(x) < t\}| < \infty$ for all $t \in [0,1)$. By truncating $(V_i)_{i\geq 0}$ and $(p_i)_{i\geq 1}$ and then applying Lemma 2.1 and the Monotone Convergence Theorem, we may first assume without loss of generality that $(V_i)_{i\geq 0}$ and $(p_i)_{i\geq 1}$ are integrable. Furthermore, we may assume that $(p_i)_{i\geq 1}$ are probability densities. For such $(V_i)_{i\geq 0}$ and $(p_i)_{i\geq 1}$, we can then apply Lemma 2.1 and the Dominated Convergence Theorem to reduce to the case where $1-\phi$ is integrable, which we assume from now on.

Let us denote $\psi := 1 - \phi$. Since $(p_i)_{i\geq 1}$ are assumed to be probability densities, we can rewrite $W_n(\phi, V_\cdot, p_\cdot)$ in (1.5) as

$$\begin{aligned} W_n(\phi, V_\cdot, p_\cdot) &= \int\cdots\int (1-\psi(x_0))\Big(1 - \prod_{i=0}^n (1 - V_i(x_i))\Big) \prod_{i=1}^n p_i(x_i - x_{i-1}) \prod_{i=0}^n \mathrm{d}x_i \\ &= -\int \psi(x_0)\,\mathrm{d}x_0 + \int\cdots\int \Big(1 - (1-\psi(x_0))\prod_{i=0}^n (1-V_i(x_i))\Big) \prod_{i=1}^n p_i(x_i - x_{i-1}) \prod_{i=0}^n \mathrm{d}x_i. \end{aligned} \tag{3.1}$$

A similar identity holds for $W_n(\phi_*, V_\cdot^*, p_\cdot^*)$, since $(p_i^*)_{i\geq 1}$ are also probability densities. We will be guided by the probabilistic interpretation that $(p_i)_{i\geq 1}$ are the transition probability densities of a random walk X, which is killed at each time $i \geq 0$ with probability $V_i(X_i)$. We can also interpret ψ as a trap function at time 0, so that X is killed at time 0 first with probability $\psi(X_0)$, and in case it survives, it is then killed with probability $V_0(X_0)$. If we start X at time 0 with Lebesgue measure, then

$$\phi_0(x_0) := (1 - V_0(x_0))(1 - \psi(x_0))$$

is the density of X_0 on $\mathbb{R}^d$ upon surviving the traps ψ and V_0. Similarly, for $n \in \mathbb{N}$,

$$\phi_n(x_n) := (1 - V_n(x_n)) \int\cdots\int (1-\psi(x_0)) \prod_{i=0}^{n-1} (1 - V_i(x_i)) \prod_{i=1}^n p_i(x_i - x_{i-1}) \prod_{i=0}^{n-1} \mathrm{d}x_i \tag{3.2}$$

is the density of X_n on $\mathbb{R}^d$ upon survival up to time n. We can then rewrite (3.1) as

$$W_n(\phi, V_\cdot, p_\cdot) + \int \psi(x_0)\,\mathrm{d}x_0 = \int (1 - \phi_n(x_n))\,\mathrm{d}x_n,$$

which is the total measure of X killed up to time n. Similarly,

$$W_n(\phi_*, V_\cdot^*, p_\cdot^*) + \int \psi^*(x_0)\,\mathrm{d}x_0 = \int (1 - \varphi_n(x_n))\,\mathrm{d}x_n,$$

where

$$\varphi_n(x_n) := (1 - V_n^*(x_n)) \int \cdots \int (1 - \psi^*(x_0)) \prod_{i=0}^{n-1} (1 - V_i^*(x_i)) \prod_{i=1}^{n} p_i^*(x_i - x_{i-1}) \prod_{i=0}^{n-1} \mathrm{d}x_i. \tag{3.3}$$

Since $\int \psi = \int \psi^*$, to prove (1.5), it then suffices to show that

$$\int (1 - \varphi_n(x_n))\,\mathrm{d}x_n \le \int (1 - \phi_n(x_n))\,\mathrm{d}x_n. \tag{3.4}$$

The key idea in proving (3.4) is to show that φ_n *symmetrically dominates* ϕ_n, denoted by $\varphi_n \succ \phi_n$, in the following sense:

$$\int_{(A^*)^c} (1 - \varphi_n(x_n))\,\mathrm{d}x_n \le \int_{A^c} (1 - \phi_n(x_n))\,\mathrm{d}x_n \qquad \text{for all measurable } A \text{ with } |A| < \infty. \tag{3.5}$$

Heuristically, this means that $\varphi_n(x)\mathrm{d}x$ contains more mass and with mass closer to ∞ than the symmetric decreasing rearrangement of $\phi_n(x)\mathrm{d}x$ around ∞. Note that $\succ$ is a partial order on the class of functions $f : \mathbb{R}^d \to [0,1]$ with $\int (1-f) < \infty$. By setting $A = \{0\}$ in (3.5), we obtain (3.4).

Note that for $n \in \mathbb{N}$,

$$\begin{aligned} \phi_n(x_n) &= (1 - V_n(x_n))(p_n * \phi_{n-1})(x_n), \\ \varphi_n(x_n) &= (1 - V_n^*(x_n))(p_n^* * \varphi_{n-1})(x_n). \end{aligned} \tag{3.6}$$

Therefore by induction, to prove $\varphi_n \succ \phi_n$, it suffices to show that: $(1-\psi^*) \succ (1-\psi)$; and if $\varphi \succ \phi$, then $(1-V^*)\varphi \succ (1-V)\phi$ and $p^* * \varphi \succ p * \phi$ for any integrable $V : \mathbb{R}^d \to [0,1]$ and any probability density $p : \mathbb{R}^d \to [0,\infty)$. The first fact holds because for any measurable A with $|A| < \infty$,

$$\int_{A^c} \psi(x)\,\mathrm{d}x = \int \psi(x)\,\mathrm{d}x - \int 1_A(x)\psi(x)\,\mathrm{d}x \ge \int \psi^*(x)\,\mathrm{d}x - \int 1_{A^*}(x)\psi^*(x) = \int_{(A^*)^c} \psi^*(x)\,\mathrm{d}x$$

by a classic rearrangement inequality (see e.g. [LL01, Theorem 3.4]). The other claims on the preservation of $\succ$ hold by Lemmas 3.2 and 3.3 below, where the integrability conditions therein are guaranteed by our integrability assumptions on ψ and $(V_i)_{i\ge 0}$. ∎

Remark 3.1 Theorem 1.2 admits two extensions which follow by the same proof as above. Firstly, (1.5) remains valid if for each $i \ge 0$, we replace $(1 - V_i(x_i))$ by $\prod_{k=1}^{l_i}(1 - V_i^{(k)}(x_i))$ for some $l_i \in \mathbb{N}$ and $V_i^{(k)} : \mathbb{R}^d \to [0,1]$ for $1 \le k \le l_i$, and replace $(1 - V_i^*(x_i))$ by $\prod_{k=1}^{l_i}(1 - V_i^{(k)*}(x_i))$. Secondly, assuming $\sigma = 1$ in Theorem 1.2 and $\int(1-\phi) < \infty$, then (1.5) also holds if we replace ϕ_* by any $\varphi : \mathbb{R}^d \to [0,1]$ such that $\int(1-\varphi) = \int(1-\phi)$, and φ symmetrically dominates ϕ in the sense defined in (3.5). This latter extension also applies to Theorem 1.4.

Lemma 3.2 *Suppose that $\phi, \varphi, V : \mathbb{R}^d \to [0,1]$ are such that $(1-\phi)$, $(1-\varphi)$ and V are all integrable. If $\varphi \succ \phi$ in the sense defined in (3.5), then we also have $(1-V^*)\varphi \succ (1-V)\phi$.*

Proof. We need to show that for all measurable A with $|A| < \infty$,

$$\int_{(A^*)^c} (1 - (1-V^*)\varphi) \le \int_{A^c} (1 - (1-V)\phi). \tag{3.7}$$

By writing $V(x) = \int_0^1 1_{F_t}(x)\,\mathrm{d}t$ with $F_t := \{x : V(x) > t\}$, and $V^*(x) = \int_0^1 1_{F_t^*}(x)\,\mathrm{d}t$, where we note that $F_t^* := \{x : V^*(x) > t\}$ is also the symmetric decreasing rearrangement of F_t, it suffices to verify

(3.7) for the case $V = 1_F$ for some measurable set F with $|F| < \infty$. For $V = 1_F$, the LHS of (3.7) equals

$$\int_{(A^*)^c} (1 - 1_{(F^*)^c}\varphi) = \int_{(A^*)^c} (1_{F^*} + 1_{(F^*)^c}(1-\varphi)) = |F^*| - |F^* \cap A^*| + \int_{(F^* \cup A^*)^c} (1-\varphi). \tag{3.8}$$

Similarly, the RHS of (3.7) equals

$$\int_{A^c} (1 - 1_{F^c}\phi) = |F| - |F \cap A| + \int_{(F \cup A)^c} (1-\phi). \tag{3.9}$$

Since $|F^*| = |F|$, and the remaining terms in (3.8) and (3.9) are symmetric in F and A, we may assume without loss of generality that $|F| \geq |A|$, which implies $A^* \subset F^*$. Subtracting (3.8) from (3.9) then gives

$$\begin{aligned} & -|F \cap A| + |A^*| + \int_{(F\cup A)^c} (1-\phi) - \int_{(F^*)^c} (1-\varphi) \\ &= |F^c \cap A| - \int_{F^c \cap A} (1-\phi) + \int_{F^c} (1-\phi) - \int_{(F^*)^c} (1-\varphi) \geq \int_{F^c \cap A} \phi \geq 0, \end{aligned} \tag{3.10}$$

where we used the fact that $|A^*| = |A|$, and the assumption $\varphi \succ \phi$. This proves (3.7). ∎

Lemma 3.3 *Suppose that $\phi, \varphi : \mathbb{R}^d \to [0,1]$ are such that $(1-\phi)$ and $(1-\varphi)$ are integrable. If $\varphi \succ \phi$, then for any probability density $p : \mathbb{R}^d \to [0,\infty)$, we have $p^* * \varphi \succ p * \phi$.*

Proof. First we note that $\varphi \succ \phi_* \succ \phi$, where $\phi_* := 1 - (1-\phi)^*$ is the symmetric increasing rearrangement of ϕ. This follows from the observation that $\varphi \succ \phi$ implies

$$\int_{(A^*)^c} (1-\varphi) \leq \inf_{B:|B|=|A|} \int_{B^c} (1-\phi) = \int_{(A^*)^c} (1-\phi)^* = \int_{(A^*)^c} (1-\phi_*) \leq \int_{A^c} (1-\phi_*).$$

For any measurable set A with $|A| < \infty$, we have

$$\begin{aligned} \int_{A^c} (1 - p*\phi) = \int_{A^c} p*(1-\phi) &= \int (1-\phi) - \iint 1_A(x) p(x-y)(1-\phi)(y)\,\mathrm{d}y\,\mathrm{d}x \\ &\geq \int (1-\phi)^* - \iint 1_{A^*}(x) p^*(x-y)(1-\phi)^*(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int (1-\phi)^*(1 - p^* * 1_{A^*}), \end{aligned}$$

where in the inequality we used Riesz's rearrangement inequality [LL01, Theorem 3.7]. Note that

$$1 - p^* * 1_{A^*}(y) = 1 - \int_0^1 1_{\{p^* * 1_{A^*}(y) > t\}}\,\mathrm{d}t = \int_0^1 1_{\{p^* * 1_{A^*}(y) \leq t\}}\,\mathrm{d}t,$$

where $\{y : p^* * 1_{A^*}(y) > t\}$, $t \in (0,1)$, are centered open balls because $p^* * 1_{A^*} = (p^* * 1_{A^*})^*$ by Lemma 2.4. Since $\varphi \succ \phi_*$, we then have

$$\begin{aligned} \int_{A^c} (1 - p*\phi) \geq \int (1-\phi)^*(1 - p^* * 1_{A^*}) &= \int_0^1 \int_{\{y: p^* * 1_{A^*}(y) \leq t\}} (1-\phi)^*(y)\,\mathrm{d}y\,\mathrm{d}t \\ &\geq \int_0^1 \int_{\{y: p^* * 1_{A^*}(y) \leq t\}} (1-\varphi)(y)\,\mathrm{d}y\,\mathrm{d}t \\ &= \int (1-\varphi)(1 - p^* * 1_{A^*}) = \int (1-\varphi) p^* * 1_{(A^*)^c} \\ &= \int 1_{(A^*)^c} p^* * (1-\varphi) = \int_{(A^*)^c} (1 - p^* * \varphi). \end{aligned}$$

Therefore $p^* * \varphi \succ p * \phi$. ∎

4 Proof of Theorem 1.4

To pave the way for the application of Theorem 1.2, we first make the following reductions.

Claim 4.1 *It is sufficient to prove Theorem 1.4 for*

(i) $\phi \in [0,1]$,

(ii) *Lévy measures* $\nu(\mathrm{d}x) = \rho(x)\mathrm{d}x$ *for some* $\rho : \mathbb{R}^d \to [0,\infty)$,

(iii) *potentials* $U : [0,\infty) \times \mathbb{R}^d \to [0,\infty)$ *which are continuous with bounded support.*

Proof of (i). This follows by the same reasoning as in the proof of Theorem 1.2. ∎

Proof of (ii). Assume that Theorem 1.4 holds under assumption (ii). Note that for a general Lévy process X with characteristic $(b, \mathbb{A}, \nu)$, with $\rho(x)\mathrm{d}x$ being the absolutely continuous part of ν, X is equally distributed with $Y + Z$, where Y is a Lévy process with characteristic $(b, \mathbb{A}, \rho(x)\mathrm{d}x)$ and $Y_0 = X_0$, and Z is an independent Lévy process with characteristic $(0, 0, \nu - \rho(x)\mathrm{d}x)$ and $Z_0 = 0$. Let $\mathbb{E}^Y_y$ denote expectation for Y with $Y_0 = y$, and let $\mathbb{E}^Z_0$ be defined similarly. By Tonelli's Theorem, we have

$$\begin{aligned}
W^X_t(\phi, U_\cdot) &= \int_{\mathbb{R}^d} \phi(x)\Big(1 - \mathbb{E}_x\Big[\exp\Big\{ - \int_0^t U_s(-X_s)\,\mathrm{d}s\Big\}\Big]\Big)\,\mathrm{d}x \\
&= \mathbb{E}^Z_0\bigg[\int_{\mathbb{R}^d} \phi(x)\Big(1 - \mathbb{E}^Y_x\Big[\exp\Big\{ - \int_0^t U_s(-Y_s - Z_s)\,\mathrm{d}s\Big\}\Big]\Big)\,\mathrm{d}x\bigg] \\
&\geq \int_{\mathbb{R}^d} \phi_*(x)\Big(1 - \mathbb{E}^{Y^*}_x\Big[\exp\Big\{ - \int_0^t U^*_s(-Y^*_s)\,\mathrm{d}s\Big\}\Big]\Big)\,\mathrm{d}x = W^{X^*}_t(\phi_*, U^*_\cdot).
\end{aligned}$$

In the inequality above, conditional on Z, we applied Theorem 1.4 for the Lévy process Y which satisfies assumption (ii), and we applied symmetric decreasing rearrangement to the potential $\tilde{U}_s(x) := U_s(x - Z_s)$. Note that because Z is a.s. càdlàg, $\tilde{U}_\cdot$ also satisfies the regularity condition (R) in Theorem 1.4; furthermore, we note that $\tilde{U}^*_\cdot = U^*_\cdot$. In the last equality above, we used the fact that Y^* and X^* are equal in law. This proves the reduction to Lévy processes satisfying (ii). ∎

Proof of (iii). We first reduce to potentials $U : [0,\infty) \times \mathbb{R}^d \to [0,\infty)$ which are bounded with bounded support. Assume that Theorem 1.4 holds for such potentials. For a general potential U satisfying the conditions in Theorem 1.4, and for each $n \in \mathbb{N}$, define $U_{n,s}(x) := 1_{s+|x|<n} U_s(x) \wedge n$. Then $U_{n,\cdot}$ is bounded with bounded support, and $U_{n,s}(x) \uparrow U_s(x)$ for all $s \geq 0$ and $x \in \mathbb{R}^d$ as $n \uparrow \infty$.

We claim that for every $x \in \mathbb{R}^d$ and for almost every realization of X with $X_0 = x$,

$$\int_0^t U_{n,s}(-X_s)\,\mathrm{d}s \uparrow \int_0^t U_s(-X_s)\,\mathrm{d}s \qquad \text{as } n \uparrow \infty. \tag{4.1}$$

Indeed, if $U_s(-X_s) < \infty$ for all $s \in [0,t)$, then (4.1) follows by the Monotone Convergence Theorem; if $U_s(-X_s) = \infty$ for some $s \in [0,t)$, so that $\int_0^t U_s(-X_s)\,\mathrm{d}s := \infty$ by our convention in (1.10), then the regularity assumption (R) in Theorem 1.4 and Remark 1.5 imply that (4.1) still holds. The Monotone Convergence Theorem then implies that

$$W^X_t(\phi, U_{n,\cdot}) \uparrow W^X_t(\phi, U_\cdot) \qquad \text{as } n \uparrow \infty. \tag{4.2}$$

If Theorem 1.4 holds for bounded potentials with bounded support, then

$$W^X_t(\phi, U_{n,\cdot}) \geq W^{X^*}_t(\phi_*, U^*_{n,\cdot}) \qquad \text{for all } n \in \mathbb{N}. \tag{4.3}$$

We claim that we also have

$$W_t^{X^*}(\phi_*, U^*_{n,\cdot}) \uparrow W_t^{X^*}(\phi_*, U^*_\cdot) \qquad \text{as } n \uparrow \infty, \tag{4.4}$$

which together with (4.2) and (4.3) will imply that $W_t^X(\phi, U_\cdot) \geq W_t^{X^*}(\phi_*, U^*_\cdot)$, and thus complete the reduction to bounded potentials with bounded support.

Indeed, by Lemma 2.1, we have $U^*_{n,s}(x) \uparrow U^*_s(x)$ for all $s \geq 0$ and $x \in \mathbb{R}^d$ as $n \uparrow \infty$. Furthermore, the potential $U^*_\cdot$ also satisfies condition (R) in Theorem 1.4. This is because $U_\cdot$ satisfies (R), which implies that its infinity level sets $(D_s)_{s \geq 0}$ satisfy

$$1_{D_s}(x) \leq \liminf_{s' \downarrow s} 1_{D_{s'}}(x) \qquad \text{for all } x \in \mathbb{R}^d, s \geq 0.$$

Therefore by Fatou's Lemma, $|D_s| \leq \liminf_{s' \downarrow s} |D_{s'}|$ for all $s \geq 0$. The assumption in Theorem 1.4 that $|\{x : U_s(x) > l\}| < \infty$ for some $l < \infty$ implies that $|D_s| < \infty$ and $\{x : U^*_s(x) = \infty\} = D^*_s$, so $|D^*_s| \leq \liminf_{s' \downarrow s} |D^*_{s'}|$. Since $(D^*_s)_{s \geq 0}$ are finite centered open balls, it is easily seen that $U^*_\cdot$ must also satisfy condition (R). The same arguments as those leading to (4.2) then imply (4.4).

We now make the reduction from bounded U with bounded support to continuous U with bounded support. For any bounded $U_\cdot(\cdot)$ with bounded support, we can find a sequence of continuous $U_{n,\cdot}(\cdot)$, uniformly bounded with uniformly bounded support, such that for all (s, x) in a set $N \subset [0, \infty) \times \mathbb{R}^d$ with full Lebesgue measure, we have $U_{n,s}(x) \to U_s(x)$ as $n \to \infty$. By Fubini's Theorem, for every realization of the Lévy process X with $X_0 = 0$, we have

$$0 = \int_0^\infty \int_{\mathbb{R}^d} 1_{N^c}(s, x)\, \mathrm{d}x\, \mathrm{d}s = \int_0^\infty \int_{\mathbb{R}^d} 1_{N^c}(s, -x - X_s)\, \mathrm{d}x\, \mathrm{d}s = \int_{\mathbb{R}^d} \int_0^\infty 1_{N^c}(s, -x - X_s)\, \mathrm{d}s\, \mathrm{d}x.$$

Therefore for every x in a set $\Lambda \subset \mathbb{R}^d$ with $|\Lambda^c| = 0$, the set $\{s \geq 0 : (s, -x - X_s) \in N\}$ has full Lebesgue measure on $[0, \infty)$. We can then write

$$\begin{aligned}
W_t^X(\phi, U_{n,\cdot}) &= \int_\Lambda \phi(x)\Big(1 - \mathbb{E}_x\Big[\exp\Big\{-\int_0^t U_{n,s}(-X_s)\, \mathrm{d}s\Big\}\Big]\Big)\, \mathrm{d}x \\
&= \mathbb{E}_0\Big[\int_\Lambda \phi(x)\Big(1 - \exp\Big\{-\int_0^t U_{n,s}(-x - X_s) 1_N(s, -x - X_s)\, \mathrm{d}s\Big\}\Big)\, \mathrm{d}x\Big] \\
&\underset{n \to \infty}{\longrightarrow} \mathbb{E}_0\Big[\int_\Lambda \phi(x)\Big(1 - \exp\Big\{-\int_0^t U_s(-x - X_s) 1_N(s, -x - X_s)\, \mathrm{d}s\Big\}\Big)\, \mathrm{d}x\Big] \\
&= \int_\Lambda \phi(x)\Big(1 - \mathbb{E}_x\Big[\exp\Big\{-\int_0^t U_s(-X_s)\, \mathrm{d}s\Big\}\Big]\Big)\, \mathrm{d}x = W_t^X(\phi, U_\cdot).
\end{aligned} \tag{4.5}$$

The convergence above holds by the Dominated Convergence Theorem because the integrands under $\mathbb{E}_0\big[\int_\Lambda \cdot\big]$ can be dominated uniformly by $1_{\{\tau_B(-x-X) < t\}}$, where B is a finite open ball containing the support of $U_{n,s}$ and U_s for all $s \geq 0$ and $n \in \mathbb{N}$, and $\tau_B(-X - x) := \inf\{s \geq 0 : -x - X_s \in B\}$. Note that

$$\mathbb{E}_0\Big[\int_\Lambda 1_{\{\tau_B(-x-X) < t\}}\, \mathrm{d}x\Big] = \int_{\mathbb{R}^d} \mathbb{P}_x(\tau_B(-X) < t)\, \mathrm{d}x,$$

which is finite by [PS71, Prop. 3.6], and hence the Dominated Convergence Theorem can be applied.

By Lemma 2.2, for Lebesgue a.e. $s \geq 0$, we have $U^*_{n,s}(x) \to U^*_s(x)$ for Lebesgue a.e. $x \in \mathbb{R}^d$. Therefore we can apply the same argument as above to conclude that $W_t^{X^*}(\phi_*, U^*_{n,\cdot}) \to W_t^{X^*}(\phi_*, U^*_\cdot)$ as $n \to \infty$. If Theorem 1.4 holds for continuous potentials with bounded support, then we have $W_t^X(\phi, U_{n,\cdot}) \geq W_t^{X^*}(\phi_*, U^*_{n,\cdot})$, which as $n \to \infty$ implies that Theorem 1.4 also holds for bounded potentials with bounded support. This concludes the reduction to potentials satisfying (iii). ∎

To prove Theorem 1.4 under the assumptions in Claim 4.1, we will follow the same steps as in [BM-H10]. We will discretize time, and approximate the Lévy process X with characteristic

$(b, \mathbb{A}, \rho(x)\mathrm{d}x)$ in the standard way by truncating its Lévy measure $\rho(x)\mathrm{d}x$, so that we have the sum of a compound Poisson process and an independent Brownian motion. For this purpose we define $\rho_n(y) := \rho(y)1_{\{|y|>1/n\}}$ and let $c_n := \int_{\mathbb{R}^d} \rho_n(y)\,\mathrm{d}y$, so that $\bar{\rho}_n(y) := c_n^{-1}\rho(y)$ is a probability density on $\mathbb{R}^d$. Let $C_{n,t}$ be a compound Poisson process, starting at 0, with characteristic function

$$\mathbb{E}_0[e^{i\langle \xi, C_{n,t}\rangle}] = e^{-t\bar{\Psi}_n(\xi)},$$

where

$$\bar{\Psi}_n(\xi) = \int_{\mathbb{R}^d} (1 - e^{i\langle \xi, y\rangle})\rho_n(y)\,\mathrm{d}y = c_n \int_{\mathbb{R}^d} (1 - e^{i\langle \xi, y\rangle})\bar{\rho}_n(y)\,\mathrm{d}y.$$

Choose ϵ_n to be a sequence of positive numbers converging to 0. Then with $\mathbb{I}_d$ denoting the $d \times d$ identity matrix, $\mathbb{A}_n := \mathbb{A} + \epsilon_n \mathbb{I}_d$ is a positive definite matrix since $\mathbb{A}$ is positive semi-definite. Let $G_{n,t}$ be a Brownian motion independent of $C_{n,t}$, starting at x, with covariance matrix $\mathbb{A}_n$ and drift $b_n = b - \int_{|y|<1} y\rho_n(y)\,\mathrm{d}y$. Now set $X_{n,t} := C_{n,t} + G_{n,t}$. Since $C_{n,t}$ and $G_{n,t}$ are independent, we get that

$$\mathbb{E}_x[e^{i\langle \xi, X_{n,t}\rangle}] = e^{-t\Psi_n(\xi) + i\langle \xi, x\rangle},$$

where

$$\Psi_n(\xi) = -i\langle b, \xi\rangle + \frac{1}{2}\langle \mathbb{A}_n \xi, \xi\rangle + \int_{\mathbb{R}^d} \big(1 + i\langle \xi, y\rangle 1_{\{|y|<1\}} - e^{i\langle \xi, y\rangle}\big)\rho_n(y)\,\mathrm{d}y.$$

We first prove a discrete time analogue of Theorem 1.4 for $X_{n,\cdot} := (X_{n,t})_{t\geq 0}$, which approximates X.

Lemma 4.2 *Let $X_{n,\cdot}$ be as above. Let $\phi : \mathbb{R}^d \to [0, \infty)$, and let $m \in \mathbb{N}$. For $1 \leq i \leq m$, let $V_i : \mathbb{R}^d \to [0, 1]$ be continuous with compact support. Let $0 < t_1 < \ldots < t_m < \infty$. Then*

$$\int_{\mathbb{R}^d} \phi(x)\Big(1 - \mathbb{E}_x\Big[\prod_{i=1}^{m}(1 - V_i(X_{n,t_i}))\Big]\Big)\,\mathrm{d}x \geq \int_{\mathbb{R}^d} \phi_*(x)\Big(1 - \mathbb{E}_x\Big[\prod_{i=1}^{m}(1 - V_i^*(X_{n,t_i}^*))\Big]\Big)\,\mathrm{d}x. \tag{4.6}$$

Proof. Let $p_{n,t}(\cdot)$ denote the transition kernel of $G_{n,t}$. With the convention that $t_0 = 0$, $k_0 = 0$, $z_0 = x$, and denoting by $N_{n,t}$ the Poisson process which counts the number of jumps of $C_{n,t}$, we can write

$$\begin{aligned} 1 - \mathbb{E}_x\Big[\prod_{i=1}^{m}(1 - V_i(X_{n,t_i}))\Big] &= \sum_{k_1\leq\ldots\leq k_m} \mathbb{P}(N_{n,t_1} = k_1, \ldots, N_{n,t_m} = k_m) \\ &\quad \times \int\cdots\int \Big(1 - \prod_{i=1}^{m}(1 - V_i(z_i))\Big) \prod_{i=1}^{m} \big(p_{n,t_i - t_{i-1}} * \bar{\rho}_n^{(k_i - k_{i-1})*}\big)(z_i - z_{i-1}) \prod_{i=1}^{m} \mathrm{d}z_i, \end{aligned} \tag{4.7}$$

where $\bar{\rho}_n^{k*}$ denotes the k-fold convolution of $\bar{\rho}_n$ with itself, and $(p_{n,t_i-t_{i-1}} * \bar{\rho}_n^{0*})(z) := p_{n,t_i-t_{i-1}}(z)$. We first rewrite (4.7) in a suitable form before applying Theorem 1.2.

On the RHS of (4.7), for each $1 \leq i \leq m$, we let $z_{i,1+k_i-k_{i-1}} := z_i$ and rewrite

$$\big(p_{n,t_i-t_{i-1}} * \bar{\rho}_n^{(k_i-k_{i-1})*}\big)(z_i - z_{i-1})\,\mathrm{d}z_i = \int\cdots\int p_{n,t_i-t_{i-1}}(z_{i,1} - z_{i-1}) \prod_{j=1}^{k_i-k_{i-1}} \bar{\rho}_n(z_{i,j+1} - z_{i,j}) \prod_{j=1}^{1+k_i-k_{i-1}} \mathrm{d}z_{i,j},$$

as well as

$$1 - V_i(z_i) = \prod_{j=1}^{1+k_i-k_{i-1}} (1 - V_{i,j}(z_{i,j})),$$

where $V_{i,j} = 0$ for all $1 \leq j \leq k_i - k_{i-1}$ and $V_{i,1+k_i-k_{i-1}} = V_i$. Integrating with respect to $\phi(z_0)\mathrm{d}z_0$, the multiple integral in (4.7) is then in the same form as the LHS of (1.5), and therefore we can apply (1.5) to obtain

$$\int\cdots\int\phi(z_0)\Big(1-\prod_{i=1}^m(1-V_i(z_i))\Big)\prod_{i=1}^m\big(p_{n,t_i-t_{i-1}}*\bar\rho_n^{(k_i-k_{i-1})*}\big)(z_i-z_{i-1})\prod_{i=0}^m\mathrm{d}z_i$$
$$\geq\int\cdots\int\phi_*(z_0)\Big(1-\prod_{i=1}^m(1-V_i^*(z_i))\Big)\prod_{i=1}^m\big(p^*_{n,t_i-t_{i-1}}*(\bar\rho_n^*)^{(k_i-k_{i-1})*}\big)(z_i-z_{i-1})\prod_{i=0}^m\mathrm{d}z_i.$$

Since $C^*_{n,\cdot}$ has the same jump rate as $C_{n,\cdot}$ with jump kernel $\bar\rho_n^*$ instead of ρ_n, and $G^*_{n,\cdot}$ has transition kernel $p^*_{n,t}$ (see e.g. [BM-H10, Sec. 3]), summing the above inequality over $0\leq k_1\leq\cdots\leq k_m$ with weights $\mathbb{P}(N_{n,t_1}=k_1,\ldots,N_{n,t_m}=k_m)$ then gives (4.6). ∎

It was shown in the proof of [BM-H10, Theorem 4.3] that $(X_{n,t_1},\ldots,X_{n,t_m})\Rightarrow(X_{t_1},\ldots,X_{t_m})$ and $(X^*_{n,t_1},\ldots,X^*_{n,t_m})\Rightarrow(X^*_{t_1},\ldots,X^*_{t_m})$ in distribution as $n\to\infty$. Using the Dominated Convergence Theorem, we can then easily extend Lemma 4.2 from $X_{n,\cdot}$ to X.

Proposition 4.3 *Let X be a Lévy process with characteristic $(b,\mathbb{A},\rho(x)\mathrm{d}x)$. Let $\phi:\mathbb{R}^d\to[0,\infty)$, and let $m\in\mathbb{N}$. For $1\leq i\leq m$, let $V_i:\mathbb{R}^d\to[0,1]$ be continuous with compact support. Let $0<t_1<\ldots<t_m<\infty$. Then*

$$\int_{\mathbb{R}^d}\phi(x)\Big(1-\mathbb{E}_x\Big[\prod_{i=1}^m(1-V_i(X_{t_i}))\Big]\Big)\,\mathrm{d}x\geq\int_{\mathbb{R}^d}\phi_*(x)\Big(1-\mathbb{E}_x\Big[\prod_{i=1}^m(1-V_i^*(X^*_{t_i}))\Big]\Big)\,\mathrm{d}x. \tag{4.8}$$

Proof of Theorem 1.4. We may assume the conditions in Claim 4.1 (i)–(iii). Since U is continuous with compact support and X is a.s. càdlàg, for every $x\in\mathbb{R}^d$ and almost surely every realization of X with $X_0=x$, we have

$$\sum_{i=1}^k\frac{t}{k}U_{it/k}(-X_{it/k})\underset{k\to\infty}{\longrightarrow}\int_0^t U_s(-X_s)\,\mathrm{d}s.$$

By the same dominated convergence argument as in (4.5), we have

$$W_t^X(\phi,U_\cdot)=\lim_{k\to\infty}\int_{\mathbb{R}^d}\phi(x)\Big(1-\mathbb{E}_x\Big[\exp\Big\{-\sum_{i=1}^k\frac{t}{k}U_{it/k}(-X_{it/k})\Big\}\Big]\Big)\,\mathrm{d}x. \tag{4.9}$$

By Lemma 2.5, U^* is also continuous with compact support. Therefore the same argument yields

$$W_t^{X^*}(\phi_*,U^*_\cdot)=\lim_{k\to\infty}\int_{\mathbb{R}^d}\phi_*(x)\Big(1-\mathbb{E}_x\Big[\exp\Big\{-\sum_{i=1}^k\frac{t}{k}U^*_{it/k}(-X^*_{it/k})\Big\}\Big]\Big)\,\mathrm{d}x. \tag{4.10}$$

Since for each $s=it/k$, we can write $e^{-U_s(-x)}=1-V_s(x)$ for a continuous $V_s:\mathbb{R}^d\to[0,1]$ with compact support, and note that $1-V_s^*(x)=e^{-U_s^*(-x)}$, we can apply Proposition 4.3 combined with (4.9)–(4.10) to obtain $W_t^X(\phi,U_\cdot)\geq W_t^{X^*}(\phi_*,U^*_\cdot)$. ∎

5 Proof of Theorem 1.9

We will derive Theorem 1.9 from Theorem 1.4.

Proof of Theorem 1.9. Let O be an open set, and recall that $T_O(X):=\inf\{s\geq 0: X_s\in O\}$. Note that applying Theorem 1.4 with $\phi\equiv 1$ and $U_s(x)=\infty\cdot 1_O(-x)$ for all $s\geq 0$ gives

$$\int_{\mathbb{R}^d}\mathbb{P}_x(T_O(X)<t)\,\mathrm{d}x\geq\int_{\mathbb{R}^d}\mathbb{P}_x(T_{O^*}(X^*)<t)\,\mathrm{d}x\qquad\text{for all } t>0. \tag{5.1}$$

We first consider $q > 0$. By Definition 1.7,

$$\begin{aligned} C_X^q(O) &= q \int_{\mathbb{R}^d} \mathbb{E}_x\big[e^{-qT_O(X)}\big]\,\mathrm{d}x = q \int_{\mathbb{R}^d} \int_0^1 \mathbb{P}_x\big(e^{-qT_O(X)} > s\big)\,\mathrm{d}s\,\mathrm{d}x \\ &= q \int_0^1 \int_{\mathbb{R}^d} \mathbb{P}_x(T_O(X) < -q^{-1}\log s)\,\mathrm{d}x\,\mathrm{d}s \\ &\geq q \int_0^1 \int_{\mathbb{R}^d} \mathbb{P}_x(T_{O^*}(X^*) < -q^{-1}\log s)\,\mathrm{d}x\,\mathrm{d}s \\ &= q \int_{\mathbb{R}^d} \int_0^1 \mathbb{P}_x\big(e^{-qT_{O^*}(X^*)} > s\big)\,\mathrm{d}s\,\mathrm{d}x = q \int_{\mathbb{R}^d} \mathbb{E}_x\big[e^{-qT_{O^*}(X^*)}\big] = C_{X^*}^q(O^*), \end{aligned} \tag{5.2}$$

where in the inequality we applied (5.1). For a general $A \in \mathcal{B}(\mathbb{R}^d)$, by (1.20), we have

$$C_X^q(A) = \inf\{C_X^q(O) : A \subset O,\, O \text{ open}\}.$$

Since for any open $O \supset A$, we have just proved that $C_X^q(O) \geq C_{X^*}^q(O^*)$, and $C_{X^*}^q(O^*) \geq C_{X^*}^q(A^*)$ by (1.19), we conclude that $C_X^q(A) \geq C_{X^*}^q(A^*)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$. This proves Theorem 1.9 for $q > 0$.

Now consider the case X is transient and $q = 0$. By Definition 1.8, for any relatively compact $A \subset \mathbb{R}^d$,

$$C_X^0(A) = \lim_{q \downarrow 0} C_X^q(A) \geq \lim_{q \downarrow 0} C_{X^*}^q(A^*) = C_{X^*}^0(A^*). \tag{5.3}$$

For general $A \in \mathcal{B}(\mathbb{R}^d)$, by definition, we have

$$C_X^0(A) := \sup\{C_X^0(K) : K \subset A,\, K \text{ relatively compact}\}.$$

Let $A_n := A \cap \{x \in \mathbb{R}^d : |x| \leq n\}$. Then $(A_n)_{n\in\mathbb{N}}$ are relatively compact, and

$$C_X^0(A) \geq C_X^0(A_n) \geq C_{X^*}^0(A_n^*).$$

Note that $(A_n^*)_{n\in\mathbb{N}}$ are finite open balls centered at the origin, and $A_n^* \uparrow A^*$ as $n \to \infty$, which implies that $C_{X^*}^0(A_n^*) \uparrow C_{X^*}^0(A^*)$ as $n \to \infty$ by (1.20) and (1.19). Therefore $C_X^0(A) \geq C_{X^*}^0(A^*)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$, which proves Theorem 1.9 for the case X is transient and $q = 0$. ∎

6 Concluding remarks

One of the open problems formulated at the end of [PS11] is the following. If X is a standard Brownian motion with $X_0 = 0$, $f : [0,\infty) \to \mathbb{R}^d$ is measurable (or even càdlàg or continuous), for which open sets D of finite volume, is the expected volume of the Wiener sausage $\bigcup_{0\leq s\leq t}(D + X_s + f(s))$ minimized when we take $f \equiv 0$? For such D, then in light of the discussion after Remark 1.5, we will call the phenomenon where the optimal path is the constant path, the *Pascal principle*. By the derivation leading to Corollary 1.6, this question is equivalent to a trap problem, where in Theorem 1.4, we take $\phi \equiv 1$, $U_s(x) = \infty \cdot 1_D(x - f(s))$, and ask whether $W_t^X(1, U_\cdot)$ is minimized at $f \equiv 0$. Note that because we are not allowed to symmetrically rearrange D, standard rearrangement inequalities will not be applicable. Generalizing from Brownian motion, we may also ask if the above Pascal principle holds for any Lévy process X whose law is equally distributed with X^*.

In light of the analogy between the random walk exit problem in (1.1) and the trap problem in (1.3), we can ask whether the Pascal principle holds for the survival probability of a Brownian motion killed upon exiting a finite domain. More precisely, let X be a standard Brownian motion (or more generally a Lévy process whose law is equally distributed with X^*), let D be a finite volume closed set with a sufficiently regular boundary, and assume that X_0 is distributed uniformly on D. Let $f : [0, \infty) \to \mathbb{R}^d$ be measurable (or even càdlàg or continuous), and let $\tau_{D^c}(X + f) := \inf\{s \geq 0 : X_s + f(s) \in D^c\}$.

$$\text{For which } D \text{ is } \quad \mathbb{P}(\tau_{D^c}(X + f) > t) \quad \text{maximized at} \quad f \equiv 0 \text{ ?}$$

When D is symmetric and convex, X is a standard Brownian motion, and f is càdlàg, the answer is affirmative and it follows from Anderson's inequality [A55] for multi-variate normal distributions (Anderson's inequality is in fact valid for general symmetric unimodal distributions). If we do not impose any assumption on the distribution of X_0, it is easily seen that the Pascal principle will fail in general. The uniform distribution on D we propose is based on the analogy with the trap problem. Another natural distribution for X_0 we may consider is the quasi-stationary distribution of X on D, which equals the limit of $\mathbb{P}(X_t \in \cdot | \tau_{D^c}(X) > t)$ as $t \to \infty$.

Acknowledgement We thank Frank Aurzada for enlightening discussions on random walk exit problems and Yuval Peres for useful remarks. R. Sun is supported by grant R-146-000-119-133 from the National University of Singapore.

References

[A55] T.W. Anderson. The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.* 6, 170–176, 1955.

[BM-H10] R. Bañuelos and P. Méndez-Hernández. Symmetrization of Lévy processes and applications. *J. Funct. Anal.* 258, 4026–4051, 2010.

[B96] J. Bertoin. *Lévy processes.* Cambridge Tracts in Mathematics, 121. Cambridge University Press.

[B04] D. Betsakos. Symmetrization, symmetric stable processes, and Riesz capacities. *Trans. Amer. Math. Soc.* 356, 735–755, 2004.

[BLL74] H.J. Brascamp, E.H. Lieb, and J.M. Luttinger. A general rearrangement inequality for multiple integrals. *J. Funct. Anal.* 17, 227–237, 1974.

[BB02] A.J. Bray and R.A. Blythe. Exact asymptotics for one-dimensional diffusion with mobile traps. *Phys. Rev. Lett.* 89, 150601, 2002.

[BS01] A. Burchard and M. Schmuckenschläger. Comparison theorems for exit times. *Geom. Funct. Anal.* 11, 651–692, 2001.

[CX11] Xia Cheng and Jie Xiong. Annealed asymptotics for Brownian motion of renormalized potential in mobile random medium. Preprint, 2011.

[DGRS10] A. Drewitz, J. Gärtner, A.F. Ramírez, and R. Sun. Survival Probability of a Random Walk Among a Poisson System of Moving Traps. Preprint, arXiv:1010.3958v3, 2010.

[FL76] R. Friedberg and J.M. Luttinger. Rearrangement inequality for periodic functions. *Arch. Ration. Mech.* 61, 35–44, 1976.

[FOT11] M. Fukushima, Y. Ōshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes.* Second revised and extended edition. De Gruyter Studies in Mathematics, 19 Walter de Gruyter & Co., Berlin, 2011.

[LL01] E.H. Lieb and M. Loss. *Analysis.* Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.

[M-H06] P.J. Méndez-Hernández. An isoperimetric inequality for Riesz capacities. *Rocky Mountain J. Math.* 36, 675–682, 2006.

[MOBC04] M. Moreau, G. Oshanin, O. Bénichou and M. Coppey. Lattice theory of trapping reactions with mobile species. *Phys. Rev. E* 69, 046101, 2004.

[PSSS11] Y. Peres, A. Sinclair, P. Sousi, and A. Stauffer. Mobile geometric graphs: Detection, coverage and percolation. *Proceedings of the 22nd ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 412–428, 2011.

[PS11] Y. Peres and P. Sousi. An isoperimetric inequality for the Wiener sausage. Preprint, arXiv:1103.6059v1, to appear in *Geom. Funct. Anal.*

[PS71] S.C. Port and C.J. Stone. Infinite divisible processes and their potential theory I. *Ann. Inst. Fourier (Grenoble)* 21, 157–275, 1971.

[S99] K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge, 1999.

[W83] T. Watanabe. The isoperimetric inequality for isotropic unimodal Lévy processes. *Z. Wahrscheinlichkeitstheorie* 63, 487–499, 1983.