TOPICS IN STATISTICAL THEORY – EXAMPLES 2/2

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Ex. 1 If $Z \sim N(0, \sigma^2)$, prove carefully that for every $u > 0$,

$$P(|Z| > u) \leq e^{-u^2/2\sigma^2}.$$

Ex. 2 (Hoeffding's inequality) Let $X_1, \dots, X_n$ be independent random variables such that

$$EX_i = 0, \;\; X_i \in [a_i, b_i], a_i < b_i, \quad \forall\; i = 1, \dots, n.$$

Prove that, for every $t > 0$,

$$\Pr\left(\sum_{i=1}^n X_i > t\right) \leq \exp\left\{-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right\}$$

and that the same bound holds for the probability of the event $\{\sum_{i=1}^n X_i < -t\}$.

Ex. 3 Let Z be a random variable satisfying the bound

$$Ee^{\lambda Z} \leq \exp\left(v(e^\lambda - 1 - \lambda)\right), \quad \lambda > 0$$

for some $v > 0$. If $h(x) = (1 + x)\log(1 + x) - x$, prove that for all $t > 0$,

$$P(Z \geq t) \leq \exp(-vh(t/v)) \leq \exp\left(-\frac{t^2}{2v + 2t/3}\right).$$

Deduce *Bernstein's inequality*: If X_i are i.i.d. centred random variables such that $|X_i| \leq c$ almost surely, then for $\sigma^2 = EX_i^2$ and every $u > 0$,

$$P\left(\sum_{i=1}^n X_i \geq u\right) \leq \exp\left(-\frac{u^2}{2n\sigma^2 + 2cu/3}\right).$$

Ex. 4 a) In the result from lectures about the hard thresholding wavelet estimator based on observations in Gaussian white noise, show that the universal choice $\tau = 5$ for the thresholding constant is admissible for the theorem to hold true.

b) Use the previous exercise to heuristically justify the construction of a hard thresholding density estimator in the problem of nonparametric density estimation on $[0, 1]$.

Ex. 5 (Nonparametric tests for uniformity) Consider $X_1, \dots, X_n$ i.i.d. random variables from density f on $[0, 1]$ contained in a Sobolev space $H^1([0, 1])$. Let $\mathbf{1} = 1_{[0,1]}$ be the density of the uniform distribution on $[0, 1]$. Consider testing the hypotheses

$$H_0 : f = \mathbf{1} \;\; vs. \;\; H_1 = \{f : \|f\|_{H^1} \leq B, \|f - \mathbf{1}\|_{L^2} \geq \rho_n\},$$

where $\rho_n = Cn^{-2/5}$ and B, C are constants. Given $\alpha > 0$, find a test Ψ such that

$$E_{\mathbf{1}}\Psi + \sup_{f \in H_1} E_f(1 - \Psi_n) \leq \alpha$$

for every $n \in \mathbb{N}$ and $C = C(\alpha)$ large enough.

Ex. 6 For a $p \times p$ symmetric matrix Φ, show that the maximal absolute eigenvalue $\phi_{\max} = \max_i |\phi_i|$ is equal to $\sup_{\|v\|_2=1} |v^T \Phi v|$. Show further that the minimal absolute eigenvalue corresponds to $\inf_{\|v\|_2=1} |v^T \Phi v|$.

Ex. 7 Let B be the unit ball in a k-dimensional Euclidean space. Let $N(\delta), \delta > 0$ be the minimal number of closed balls of radius δ with centers in B that are required to cover B. Show that for some constant $A > 0$ and every $0 < \delta < A$ we have

$$N(\delta) \leq \left(\frac{A}{\delta}\right)^k .$$