TOPICS IN STATISTICAL THEORY – EXAMPLES 1/2

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Ex. 1 In what follows $L^2([0,1])$ denotes the space of functions $f:[0,1]\to\mathbb{R}$ such that

$$\|f\|^2_{L^2([0,1])} \equiv \int_0^1 f^2(x)dx < \infty,$$

where dx is Lebesgue measure on $[0,1]$. Moreover $L^2(P)$ denotes the space of random variables X defined on the probability space $(\Omega,\mathcal{A},P)$ such that $EX^2 \equiv \|X\|^2_{L^2(P)} < \infty$.

a) [Existence of the *Wiener* and the *Gaussian white noise* process]. Show that the mappings $\Phi: T\times T\to\mathbb{R}$ given by

$$\Phi(s,t)=\min(s,t), \quad T=[0,1],$$

and by

$$\Phi(s,t)=\int_0^1 s(x)t(x)dx, \quad T=L^2([0,1]),$$

define *covariances* of a Gaussian process $(G(t):t\in T)$.

b) [Construction of Gaussian white noise as a *stochastic integral*.] Let $(W(t):t\in[0,1])$ be a Wiener process defined on some probability space $(\Omega,\mathcal{A},P)$. For $h\in L^2([0,1])$ take approximations $h_N=\sum_{i\le N} a_i 1_{I_i}$ where the $I_i=(x_i,y_i]$'s are disjoint intervals, and where the a_i's are real coefficients, such that $\|h_N-h\|_{L^2([0,1])}\to 0$ as $N\to\infty$. Define new random variables

$$\mathbb{W}(1_{(x,y]}) = W(y)-W(x), x<y, \text{ and } \mathbb{W}(h_N)=\sum_{i\le N} a_i \mathbb{W}(1_{I_i}).$$

Show that the $L^2(P)$-limit $\mathbb{W}(h)$ of $\mathbb{W}(h_N)$ exists, and that $(\mathbb{W}(h):h\in L^2([0,1])$ is a Gaussian process with covariance $E\mathbb{W}(g)\mathbb{W}(h)=\int_0^1 g(x)h(x)dx$ for all $g,h\in L^2([0,1])$.

c) Consider an infinite sequence of random variables $(g_k:k\in\mathbb{N})$, all i.i.d. $N(0,1)$, defined on some common probability space $(\Omega,\mathcal{A},P)$. For $\{e_k:k\in\mathbb{N}\}$ any orthonormal basis of $L^2([0,1])$, show that $\mathbb{W}(h)$ from b) can also be obtained as the limit in $L^2(P)$ of the random variables $\tilde{\mathbb{W}}_N=\sum_{k\le N} h_k g_k$ where $h_k=\int_0^1 h(x)e_k(x)dx$.

Ex. 2 Let P,Q be two probability measures on a measure space $(\mathcal{X},\mathcal{A})$. Suppose there exists a common dominating measure μ on $(\mathcal{X},\mathcal{A})$, and let $p=dP/d\mu, q=dQ/d\mu$. Show that

$$\sup_{A\in\mathcal{A}} |P(A)-Q(A)| = \frac{1}{2}\int_{\mathcal{X}} |p(x)-q(x)|d\mu(x).$$

Ex. 3 Let $N\in\mathbb{N}$ and let $\{x_0,x_1,\dots,x_N\}$ be the dissection of $[0,1]$ given by $x_i=i/N, i=0,\dots,N$. Denote by $\mathcal{V}_N$ the space of functions that are piece-wise constant on the intervals $I_{iN}=(x_{i-1},x_i], i=1,\dots N$. Show that

a) the mapping $(f,g)\mapsto\langle f,g\rangle_N=\sum_{i=1}^N f(x_i)g(x_i)$ defines an inner product on $\mathcal{V}_N$,

b) the functions $\{1_{I_{iN}}:i=1,\dots N\}$ form an orthonormal basis of $\mathcal{V}_N$ for $\langle\cdot,\cdot\rangle_N$,

c) the function

$$f_N=\sum_{i=1}^N \langle f,1_{I_{iN}}\rangle_N 1_{I_{in}} \in \mathcal{V}_N$$

interpolates a function f at the points x_i (that is, it satisfies $f_N(x_i) = f(x_i)$ for all $i = 1, \dots, N$),

d) the functions

$$\{\phi_{iN} \equiv \sqrt{n} 1_{I_{iN}} : i = 1, \dots N\}$$

form an orthonormal basis of $\mathcal{V}_N$ for the inner product $\langle \cdot, \cdot \rangle_{L^2}$ of $L^2([0,1])$,

e) the $L^2([0,1])$-projection of $f \in L^2$ onto $\mathcal{V}_N$,

$$\pi_N(f) = \sum_{i=1}^{N} \langle f, \phi_{iN} \rangle_{L^2} \phi_{iN},$$

satisfies, for any f of α-Hölder norm $\|f\|_{C^\alpha} \leq M, 0 < \alpha \leq 1$, the bound

$$\|f - \pi_N(f)\|_{L^2}^2 \leq \|f - f_N\|_{L^2}^2 \lesssim M^2 N^{-2\alpha}.$$

Ex. 4 Let $\alpha \in \mathbb{N}$. Consider a function $f \in L^2([0,1])$ such that $D^m f \in L^2, D^m f(0) = D^m f(1)$, for all $0 \leq m \leq \alpha$. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product of $L^2([0,1])$ and consider the trigonometric basis of $L^2([0,1])$ given by $\{e_k = e^{-2\pi i k \cdot} : k \in \mathbb{Z}\}$. Show that

$$\sum_{k \in \mathbb{Z}} |\langle f, e_k \rangle|^2 (1 + k^{2\alpha}) < \infty.$$

If $K \in \mathbb{N}$ and $f_K = \sum_{|k| \leq K} \langle f, e_k \rangle e_k$, deduce that $\|f_K - f\|_{L^2([0,1])}^2 \leq K^{-2\alpha}$.

Ex. 5 Suppose two probability measures P, Q have a common dominating measure μ on a measure space $(\mathcal{X}, \mathcal{A})$, and are absolutely continuous with respect to each other. Write $p = dP/d\mu$ and $q = dQ/d\mu$, and denote by $K(P,Q)$ the Kullback-Leibler divergence between P and Q. Show that

$$E_P \left| \log \frac{p}{q}(X) \right| \leq K(P,Q) + \sqrt{2K(P,Q)}$$

Ex. 6 Let $X_1, \dots, X_n$ be i.i.d. random variables with bounded density $f : [0,1] \to [0, \infty)$. Assuming that f is also α-Hölder for some $0 < \alpha \leq 1$, show that one can construct an estimator $\hat{f}_n$ of f such that

$$E_f \|\hat{f}_n - f\|_{L^2([0,1])}^2 \leq C n^{-2\alpha/(2\alpha+1)},$$

where $C > 0$ is a constant independent of sample size. [Hint: Estimate the Haar wavelet approximation $\Pi_{V_J}(f)$ of f at resolution level J unbiasedly from the sample, and choose J to 'balance bias and variance'.]

Ex. 7 In the setting of the previous exercise with $\alpha = 1$, show carefully that the rate of convergence obtained is minimax optimal over the class

$$\mathcal{F} = \left\{ f : [0,1] \to [0,B], \ \int_0^1 f(x)dx = 1, \ \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|} \leq B \right\}, \quad B > 1,$$

of probability density functions that are Lipschitz (i.e., α-Hölder with $\alpha = 1$).