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Example Sheet 4 - Michaelmas 2006

Problem 1. Let $Y_1, Y_2, \ldots$ be independent, identically distributed random variables with values in $\{1, 2, \ldots\}$. Suppose that the set of integers

$$\{n : \mathbb{P}(Y_1 = n) > 0\}$$

has greatest common divisor 1. Set $\mu = \mathbb{E}(Y_1)$. Prove that the following process is a Markov chain: $X_0 = 0$ and

$$X_n = \inf\{m \geq n : m = Y_1 + \ldots + Y_k \text{ for some } k \geq 1\} - n$$

for $n \geq 1$. Furthermore show that

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0) = \frac{1}{\mu}.$$

Problem 2. Consider a continuous time Markov chain with generator

$$G = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}.$$

Use the Kolmogorov equations to find the transition probabilities. What is the invariant distribution?

Problem 3. Consider a continuous time immigration-death process with constant immigration rate $\lambda_i = \lambda$ and proportional death rate $\mu_i = i\mu$. That is, the generator $G = (g_{ij})_{i,j\geq 0}$ is such that

$$g_{ij} = \begin{cases} \lambda & \text{if } j = i+1 \\ -(\lambda + i\mu) & \text{if } j = i \\ i\mu & \text{if } j = i-1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that the invariant distribution is Poisson with parameter λ/μ.

Problem 4. Let $(X_n)_n$ be a Markov chain on S with transition matrix $P = (p_{ij})_{(i,j)\in S\times S}$. Consider a bounded function $f : S \to \mathbb{R}$ such that

$$\sum_{j\in S} p_{ij} f(j) = f(i).$$

for all states $j \in S$. Prove that $(f(X_n))_n$ is a martingale with respect to the filtration generated by $(X_n)_n$.

Problem 5. Let $X_1, X_2, \ldots$ be independent and identically distributed with common moment generating function $M(t) = \mathbb{E}(e^{tX_1})$, and let $S_n = X_1 + \ldots + X_n$. Show that

$$Z_n = e^{tS_n} M(t)^{-n}$$

is a martingale with respect to the filtration generated by $(X_n)_n$ for all t for which the moment generating function is finite.

Problem 6. Let $X_1, X_2, \ldots$ be independent and identically distributed with

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2},$$

and let $S_n = X_1 + \ldots + X_n$. Fix a natural number k and define the random time

$$\tau = \inf\{n \geq 1 : |S_n| = k\}.$$

Use Problem 5 and the optional stopping theorem to show that the probability generating function $G(s) = \mathbb{E}(s^\tau)$ is given by

$$G(s) = \operatorname{sech}(k \operatorname{sech}^{-1}(s))$$

for $0 < s \leq 1$. Recall that

$$\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}.$$

Problem 7. Let $(M_n)_{n\geq 0}$ be a bounded martingale. The goal of this exercise is to prove Doob's maximal inequality.

1. Use Jensen's inequality to show that $(|M_n|)_n$ is a submartingale.
2. Fix $\lambda \geq 0$ and let $\tau = \inf\{n \geq 0 : |M_n| \geq \lambda\}$. By using the optional sampling theorem at the stopping times $\tau \wedge N$ and N, show that

   $$\lambda \mathbb{P}(M^* \geq \lambda) \leq \mathbb{E}(|M_N| \mathbb{1}_{\{M^* \geq \lambda\}})$$

   where $M^* = \max_{0\leq n\leq N} |M_n|$.
3. Integrate both sides with respect to λ to show

   $$\mathbb{E}[(M^*)^2] \leq 2\, \mathbb{E}(|M_N| M^*)$$

4. Use the Cauchy-Schwarz inequality to prove

   $$\mathbb{E}(\max_{0\leq n\leq N} M_n^2) \leq 4\, \mathbb{E}(M_N^2).$$