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Example Sheet 4 - Michaelmas 2006

Problem 1. Let $Y_1, Y_2, \ldots$ be independent, identically distributed random variables with values in $\{1, 2, \ldots\}$. Suppose that the set of integers

$$\{n : \mathbb{P}(Y_1 = n) > 0\}$$

has greatest common divisor 1. Set $\mu = \mathbb{E}(Y_1)$. Prove that the following process is a Markov chain: $X_0 = 0$ and

$$X_n = \inf\{m \geq n : m = Y_1 + \ldots + Y_k \text{ for some } k \geq 1\} - n$$

for $n \geq 1$. Furthermore show that

$$\lim_{n\uparrow\infty} \mathbb{P}(X_n = 0) = \frac{1}{\mu}.$$

Solution 1. Let $S_k = Y_1 + \ldots + Y_k$ for $k \geq 1$, and let

$$K_n = \inf\{k \geq 1 : S_k \geq n\}$$

so that

$$X_n = S_{K_n} - n.$$

Since

$$X_{n+1} = \begin{cases} X_n - 1 & \text{if } X_n \geq 1 \\ Y_{K_n+1} - 1 & \text{if } X_n = 0 \end{cases}$$

and the random variable Y_{K_n+1} is independent of $X_1, \ldots, X_n$, it follows that $(X_n)_n$ is a time homogeneous Markov chain with transition probabilities

$$p_{0,j} = \mathbb{P}(Y_1 = j + 1) \text{ and } p_{j+1,j} = 1 \text{ for all } j \geq 0.$$

The chain is irreducible since all states communicate with 0. The first time that the chain visits 0 is $\inf\{n \geq 1 : X_n = 0\} = Y_1$. Because $\mathbb{E}(Y_1) = \mu < +\infty$ the chain is positive recurrent, and hence has a unique invariant measure π satisfying $\pi_0 = 1/\mu$.

[Alternatively, one could solve the equation $\pi P = \pi$ to deduce

$$\pi_i = \frac{\mathbb{P}(Y_1 > i)}{\mu}.$$

where $P = (p_{ij})_{ij}$.]

Since

$$\mathbb{P}(X_{n+m} = 0 | X_m = 0) = \mathbb{P}(Y_1 = n)$$

the set of n's such that the n-step transition probabilities are positive have greatest common divisor 1, and hence the chain is aperiodic. Therefore, the ergodic theorem implies

$$\lim_{n\uparrow\infty} \mathbb{P}(X_n = 0) = \pi_0 = \frac{1}{\mu}.$$

Problem 2. Consider a continuous time Markov chain with generator

$$G = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}.$$

Use the Kolmogorov equations to find the transition probabilities. What is the invariant distribution?

Solution 2. The forward Kolmogorov equation is

$$P'(t) = P(t)G, \; P(0) = I$$

or in component form:

$$\begin{cases} p'_{11}(t) &= -\mu p_{11}(t) + \lambda p_{12}(t), \quad p_{11}(0) = 1 \\ p'_{12}(t) &= \mu p_{11}(t) - \lambda p_{12}(t), \quad p_{12}(0) = 0 \\ p'_{21}(t) &= -\mu p_{21}(t) + \lambda p_{22}(t), \quad p_{21}(0) = 0 \\ p'_{22}(t) &= \mu p_{21}(t) - \lambda p_{22}(t), \quad p_{22}(0) = 1 \end{cases}$$

The solution is

$$\begin{cases} p_{11}(t) &= \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-t(\lambda+\mu)} \\ p_{12}(t) &= \frac{\mu}{\lambda+\mu}(1 - e^{-t(\lambda+\mu)}) \\ p_{21}(t) &= \frac{\lambda}{\lambda+\mu}(1 - e^{-t(\lambda+\mu)}) \\ p_{22}(t) &= \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-t(\lambda+\mu)} \end{cases}$$

Alternatively, one could solve the backward Kolmogorov equation

$$P'(t) = GP(t), \; P(0) = I.$$

In either case, the solution is given by $P(t) = e^{tG}$. The matrix exponential can also be calculated from the decomposition

$$G = \begin{pmatrix} 1 & -\mu \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -(\lambda+\mu) \end{pmatrix} \begin{pmatrix} 1 & -\mu \\ 1 & \lambda \end{pmatrix}^{-1}$$

as

$$e^{tG} = \begin{pmatrix} 1 & -\mu \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-t(\lambda+\mu)} \end{pmatrix} \begin{pmatrix} 1 & -\mu \\ 1 & \lambda \end{pmatrix}^{-1}.$$

The invariant distribution can either be found by letting $t \uparrow \infty$ in the above equations:

$$\pi_1 = \frac{\lambda}{\lambda+\mu}, \quad \pi_2 = \frac{\mu}{\lambda+\mu}$$

or by solving $\pi G = 0$ subject to $\pi_1 + \pi_2 = 1$.

Problem 3. Consider a continuous time immigration-death process with constant immigration rate $\lambda_i = \lambda$ and proportional death rate $\mu_i = i\mu$. That is, the generator $G = (g_{ij})_{i,j\geq 0}$ is such that

$$g_{ij} = \begin{cases} \lambda & \text{if } j = i+1 \\ -(\lambda + i\mu) & \text{if } j = i \\ i\mu & \text{if } j = i-1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that the invariant distribution is Poisson with parameter λ/μ.

Solution 3. Let π be the row vector with entries

$$\pi_i = \frac{(\lambda/\mu)^i}{i!} e^{-\lambda/\mu}.$$

Since

$$\begin{aligned} (\pi G)_j &= \begin{cases} \pi_0 g_{00} + \pi_1 g_{10} & \text{if } j = 0 \\ \pi_{j-1} g_{j-1,j} + \pi_j g_{jj} + \pi_{j+1} g_{j,j+1} & \text{if } j > 0 \end{cases} \\ &= 0 \end{aligned}$$

the measure π is invariant.

Problem 4. Let $(X_n)_n$ be a Markov chain on S with transition matrix $P = (p_{ij})_{(i,j)\in S\times S}$. Consider a bounded function $f : S \to \mathbb{R}$ such that

$$\sum_{j\in S} p_{ij} f(j) = f(i).$$

for all states $j \in S$. Prove that $(f(X_n))_n$ is a martingale with respect to the filtration generated by $(X_n)_n$.

Solution 4. If $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ then

$$\mathbb{E}[f(X_{n+1})|\mathcal{F}_n] = \mathbb{E}[f(X_{n+1})|X_n]$$

by the Markov property. But

$$\mathbb{E}[f(X_{n+1})|X_n = i] = \sum_{j\in S} p_{ij} f(j) = f(i)$$

by assumption so that

$$\mathbb{E}[f(X_{n+1})|\mathcal{F}_n] = f(X_n).$$

Problem 5. Let $X_1, X_2, \ldots$ be independent and identically distributed with common moment generating function $M(t) = \mathbb{E}(e^{tX_1})$, and let $S_n = X_1 + \ldots + X_n$. Show that

$$Z_n = e^{tS_n} M(t)^{-n}$$

is a martingale with respect to the filtration generated by $(X_n)_n$ for all t for which the moment generating function is finite.

Solution 5.

$$\mathbb{E}(Z_{n+1}|\mathcal{F}_n) = \mathbb{E}(e^{tX_{n+1}}M(t)^{-1}Z_n|\mathcal{F}_n) = \mathbb{E}(e^{tX_{n+1}})M(t)^{-1}Z_n = Z_n$$

Problem 6. Let $X_1, X_2, \ldots$ be independent and identically distributed with

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2},$$

and let $S_n = X_1 + \ldots + X_n$. Fix a natural number k and define the random time

$$\tau = \inf\{n \geq 1 : |S_n| = k\}.$$

Use Problem 5 and the optional stopping theorem to show that the probability generating function $G(s) = \mathbb{E}(s^\tau)$ is given by

$$G(s) = \operatorname{sech}(k \operatorname{sech}^{-1}(s))$$

for $0 < s \leq 1$. Recall that

$$\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}.$$

Solution 6. The moment generating function of X_1 is given by $\mathbb{E}(e^{tX_1}) = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh t$. By the previous exercise, the process

$$Z_n(t) = e^{tS_n}(\cosh t)^{-n}$$

defines a martingale for every $t \in \mathbb{R}$. Since the random variables $|S_{\tau\wedge n}|$ are bounded by k and $\cosh t \geq 1$, we have $Z_n(t) \leq e^{kt}$ for all n and hence the the optional stopping theorem implies

$$\mathbb{E}(Z_\tau(t)) = 1.$$

On the other hand,

$$\begin{aligned}\frac{1}{2}\mathbb{E}[Z_\tau(t) + Z_\tau(-t)] &= \mathbb{E}[\cosh(tS_\tau)(\cosh t)^{-\tau}] \\ &= \cosh(kt)\mathbb{E}[(\cosh t)^{-\tau}]\end{aligned}$$

since $\cosh(tS_\tau) = \cosh(tk)$ on the almost sure event $\{\tau < \infty\}$.

Letting $t = \operatorname{sech}^{-1} s$ and rearranging completes the proof.

Problem 7. Let $(M_n)_{n\geq 0}$ be a bounded martingale. The goal of this exercise is to prove Doob's maximal inequality.

1. Use Jensen's inequality to show that $(|M_n|)_n$ is a submartingale.

2. Fix $\lambda \geq 0$ and let $\tau = \inf\{n \geq 0 : |M_n| \geq \lambda\}$. By using the optional sampling theorem at the stopping times $\tau \wedge N$ and N, show that

$$\lambda\mathbb{P}(M^* \geq \lambda) \leq \mathbb{E}(|M_N|\mathbb{1}_{\{M^*\geq\lambda\}})$$

where $M^* = \max_{0\leq n\leq N} |M_n|$.

3. Integrate both sides with respect to λ to show

$$\mathbb{E}[(M^*)^2] \leq 2\ \mathbb{E}(|M_N|M^*)$$

4. Use the Cauchy-Schwarz inequality to prove

$$\mathbb{E}(\max_{0\leq n\leq N} M_n^2) \leq 4\ \mathbb{E}(M_N^2).$$

Solution 7. 1. $\mathbb{E}(|M_{n+1}|\ \big|\mathcal{F}_n) \geq |\mathbb{E}(M_{n+1}|\mathcal{F}_n)| = |M_n|$

2. Since $\tau \wedge N \leq N$ and $(|M_n|)_n$ is a submartingale, we have by the optional sampling theorem

$$\mathbb{E}|M_{\tau\wedge N}| \leq \mathbb{E}|M_N|.$$

We have $|M_{\tau\wedge N}| = \lambda \mathbb{1}_{\{\tau\leq N\}} + |M_N|\mathbb{1}_{\{\tau>N\}}$. But since $\{\tau \leq N\} = \{M^* \geq \lambda\}$, we have

$$\mathbb{E}|M_{\tau\wedge N}| = \lambda\mathbb{P}(M^* \geq \lambda) + \mathbb{E}(|M_N|\mathbb{1}_{\{M^*<\lambda\}}).$$

Hence

$$\begin{aligned}\lambda\mathbb{P}(M^* \geq \lambda) &= \mathbb{E}|M_{\tau\wedge N}| - \mathbb{E}(|M_N|\mathbb{1}_{\{M^*<\lambda\}}) \\ &\leq \mathbb{E}|M_N| - \mathbb{E}(|M_N|\mathbb{1}_{\{M^*<\lambda\}}) = \mathbb{E}(|M_N|\mathbb{1}_{\{M^*\geq\lambda\}})\end{aligned}$$

3. Integrate both sides of the above inequality with respect to λ. On the left side we have $\int_0^\infty \lambda\mathbb{P}(M^* \geq \lambda)d\lambda = \frac{1}{2}\mathbb{E}[(M^*)^2]$.

On the ride side, note that $\int_0^\infty \mathbb{1}_{\{M^*\geq\lambda\}}d\lambda = M^*$ identically. The interchange of expectation and integration with respect to λ is justified since the integrand is almost surely positive.

4. $$\mathbb{E}[(M^*)^2] \leq 2\mathbb{E}(|M_N|M^*) \leq 2\left(\mathbb{E}[M_N^2]\right)^{1/2}\left(\mathbb{E}[(M^*)^2]\right)^{1/2}$$

by the Cauchy-Schwarz inequality. Squaring both sides and dividing by $\mathbb{E}(M^*)^2$ yields the desired conclusion.