Introduction to Probability Michael Tehranchi
Example Sheet 3 - Michaelmas 2006

Problem 1. Let $(X_n)_n$ be a homogeneous Markov chain on S with transition matrix P. Given a $k \in \mathbb{N}$, let $Z_n = X_{kn}$. Prove that $(Z_n)_n$ is Markov chain with transition matrix P^k.

Problem 2. Let $U_1, U_2, \ldots$ be a sequence of independent random variable uniformly distributed on $[0,1]$. Given a function $G : S \times [0,1] \to S$, let $(X_n)_n$ be defined recursively by

$$X_{n+1} = G(X_n, U_{n+1}).$$

Show that $(X_n)_n$ is a Markov chain on S. Prove that all Markov chains can be realized in this fashion with a suitable choice of the function G.

Problem 3. Let $X_1, X_2, \ldots$ be a sequence of independent random variables with

$$\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = \frac{1}{2}$$

for all n. Let $S_n = X_1 + \ldots + X_n$. Prove that $(S_n)_{n \geq 0}$ is a recurrent Markov chain on the set of integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

[You might find Stirling's formula useful: $n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n}$.]

Problem 4. Fix a natural number $d \geq 1$, and let $X_1, X_2, \ldots$ be a sequence of independent vector-valued random variables with

$$\mathbb{P}(X_n = e_i) = \mathbb{P}(X_n = -e_i) = \frac{1}{2d}$$

for all $1 \leq i \leq d$, where

$$e_i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{d-i})$$

Let $S_n = X_1 + \cdots + X_n$. Prove that $(S_n)_{n \geq 0}$ is a transient Markov chain on $\mathbb{Z}^d$ if and only if $d \geq 3$.

[You may find the following inequality useful: If $i_1 + \ldots + i_d = dn$ then $i_1! \cdots i_d! \geq (n!)^d$.]

Problem 5. Let T_1 and T_2 be stopping times for a Markov chain $(X_n)_{n \geq 0}$ on S. Prove that each of the following are also stopping times:

1. $T = \min\{n \geq 1 : X_n = i\}$ for some fixed $i \in S$.
2. $T(\omega) = N$ for all $\omega \in \Omega$ for a fixed $N \in \mathbb{N}$.
3. $T = \min\{T_1, T_2\}$.
4. $T = \max\{T_1, T_2\}$.
5. $T = T_1 + T_2$.

Problem 6. A flea hops randomly on the vertices of a triangle with vertices labelled 1,2, and 3, hopping to each of the other vertices with equal probability. If the flea starts at vertex 1, find the probability that after n hops the flea is back to vertex 1.

A second flea also starts at vertex 1 and hops about on the vertices of a triangle, but this flea is twice as likely to jump clockwise as anticlockwise. What is the probability that after n hops this second flea is back to vertex 1?

Problem 7. Consider the second flea of Problem 6. What is the expected number of hops the flea makes before it is first back to vertex 1? What is the expected number of times the flea visits vertex 3 before first reaching vertex 2? (Assume that the vertices are labelled so that $1 \to 2 \to 3 \to 1 \ldots$ is clockwise.)