Introduction to Probability Michael Tehranchi

Example Sheet 2 - Michaelmas 2006

Problem 1. Let $X_1, X_2, \ldots$ be independent and identically distributed exponential random variables with parameter λ. Prove

$$\limsup_{n\uparrow\infty} \frac{X_n}{\log n} = \frac{1}{\lambda}$$

almost surely.

Problem 2. Let $X_1, X_2, \ldots$ be independent absolutely continuous random variables such that X_n has density function f_n given by

$$f_n(x) = \frac{n}{\pi(1+n^2x^2)}.$$

With respect to which modes of convergence does X_n converge to zero as $n \uparrow \infty$?

Problem 3. Let $M(t) = \mathbb{E}(e^{tX})$ be the moment generating function of a random variable X. Prove

$$\mathbb{P}(X \geq \epsilon) \leq \inf_{t\geq 0} e^{-\epsilon t} M(t).$$

Problem 4. Let X and Y be jointly normal with zero means and unit variances and correlation ρ. Prove

$$\mathbb{P}(X > 0, Y > 0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho.$$

Problem 5. If $X_n \to X$ in L_1 then prove $\mathbb{E}(X_n) \to \mathbb{E}(X)$. If $X_n \to X$ in L_2 then prove $\mathrm{Var}(X_n) \to \mathrm{Var}(X)$.

Problem 6. Let $X_1, X_2, \ldots$ be a sequence of random variables such that $\mathbb{E}\sup_{n\geq 1}|X_n| < \infty$. If $X_n \to X$ in probability, then prove $X_n \to X$ in L_1.

Problem 7. Let $X_2, X_3, \ldots$ be a sequence of independent random variables such that

$$\mathbb{P}(X_n = n) = \mathbb{P}(X_n = -n) = \frac{1}{2n\log n}; \quad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n\log n}.$$

Let $S_n = X_2 + \ldots + X_n$. Prove that $\frac{S_n}{n} \to 0$ in probability, but not almost surely. (That is, the weak law of large numbers holds, but not the strong law.)

Problem 8. Let X be a random variable and let $K_X(t) = \log\mathbb{E}(e^{tX})$ be the logarithm of the moment generating function. Prove that K_X is convex. Suppose that K_X has a Taylor series

$$K_X(t) = \sum_{n=1}^{\infty} \frac{1}{n!} k_n(X) t^n.$$

Compute $k_1(X), k_2(X)$, and $k_3(X)$ in terms of the moments of X. If X and Y are independent, prove that $k_n(X+Y) = k_n(X) + k_n(Y)$ for all $n \geq 1$.

Problem 9. Let $X_1, X_2, \ldots$ be a sequence of independent random variables each uniformly distributed uniformly $[0, 1]$. Let $M_n = \min\{X_1, \ldots, X_n\}$. Prove that nM_n converges in distribution to an exponential random variable with parameter one.

Problem 10. If a sequence of random variables $X_n \to c$ in distribution, where c is a constant, then prove $X_n \to c$ in probability.

Problem 11. Let $X_n \to X$ in probability. Prove that there exists a subsequence $n_1 < n_2 < \ldots$ such that $X_{n_k} \to X$ almost surely.

Problem 12. Let U be uniformly distributed on $[0, 1]$. Let the conditional distribution of X given U be binomial with parameters U and n. Find the distribution of X.

Problem 13. The random variables X and Y are distributed uniformly on the disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ Find the density of the random variable X/Y.