Introduction to Probability Michael Tehranchi

Example Sheet 1 - Michaelmas 2006

Problem 1. Show that if $\mathcal{F}$ is a sigma-field on a set Ω then that both Ω and $\emptyset$ are elements of $\mathcal{F}$.

Problem 2. Let $\mathcal{F}$ and $\mathcal{G}$ be two sigma-fields on a set Ω. Prove that $\mathcal{F} \cap \mathcal{G}$ is also a sigma-field on Ω. Show by example that $\mathcal{F} \cup \mathcal{G}$ may fail to be a sigma-field.

Problem 3. Let $A_1, A_2, A_3, \ldots$ be a sequence of events such that $A_1 \subset A_2 \subset A_3 \subset \ldots$, and let $A = \bigcup_{n=1}^{\infty} A_n$. Prove that

$$\mathbb{P}(A) = \lim_{n \uparrow \infty} \mathbb{P}(A_n).$$

Problem 4. Let X be a real-valued random variable, and let F_X be its distribution function. Show that

$$\lim_{t \downarrow -\infty} F_X(t) = 0, \quad \lim_{t \uparrow +\infty} F_X(t) = 1, \quad \mathbb{P}(X = t) = F_X(t) - \lim_{s \uparrow t} F_X(s).$$

In particular, the distribution function of X is defective if and only if

$$\mathbb{P}(X = +\infty \text{ or } X = -\infty) > 0.$$

Problem 5. Let $F : \mathbb{R} \to [0,1]$ be increasing, right-continuous, with

$$\lim_{t \downarrow -\infty} F(t) = 0 \text{ and } \lim_{t \uparrow +\infty} F(t) = 1.$$

Show that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there is a random variable X such that F is the distribution function of X. (Hint: Let the sample space Ω be the unit interval $[0,1]$, the events $\mathcal{F}$ the Borel sigma-field, and $\mathbb{P}$ the uniform measure.)

Problem 6. A box contains 20 balls, labelled with the numbers from 1 to 20. Three balls are drawn at random from the box. Find the probability that 10 is the smallest label of the three balls.

Problem 7. A coin is tossed n times with probability p of heads on each toss. Let E be the event that the first toss lands heads. Let F be the event that there are exactly k heads. For which pairs of natural numbers n and k are the events E and F independent?

Problem 8. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables with mean μ and variance σ^2. Find the mean of the random variables $\bar{X}$ and S^2 where

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \text{ and } S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

If $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, prove that $\bar{X}$ and S^2 are independent.

Problem 9. A population contains n animals. Every day, an animal is captured at random. If that animal has not been captured before, it is tagged and released. Prove that the expected number of days needed to tag all the animals is $n \sum_{i=1}^{n} 1/i$.

Problem 10. Let X and Y be independent Poisson random variables with parameters λ and μ respectively. What is the conditional distribution of X given $X + Y = n$?

Problem 11. Let X and Y be independent geometric random variables with the same parameter p. What is the conditional distribution of X given $X + Y = n$?

Problem 12. Let X be a geometric random variable. Prove that $\mathbb{P}(X = n + m | X > m) = \mathbb{P}(X = n)$ for all $m, n = 1, 2, 3, \ldots$. Why does one say that geometric random variables are memoryless?

Problem 13. Let X be an exponential random variable. Prove that $\mathbb{P}(X > t + s | X > s) = \mathbb{P}(X > t)$ for all $s, t \geq 0$. Why does one say that exponential random variables are memoryless?

Problem 14. Let $X_1, \ldots, X_n$ be uncorrelated random variables with mean 0 and variance 1. Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. By using Chebyshev's inequality, find the smallest n such that

$$\mathbb{P}(|\bar{X}| > 2) \leq \frac{1}{100}.$$

Now redo this problem under the additional assumption that the random variables $X_1, \ldots, X_n$ are jointly normal.

Problem 15. Let X and Y be independent exponential random variables with parameters λ and μ respectively. Show that $U = \min\{X, Y\}$ is exponential with parameter $\lambda + \mu$. Show that the events $\{U \leq t\}$ and $\{X < Y\}$ are independent for all t.

Problem 16. Let X and Y be identically distributed random variables with finite mean and variance. Show that $U = X + Y$ and $V = X - Y$ are uncorrelated. Show by example that U and V need not be independent.

Problem 17. Let $X \sim N(\mu, \sigma^2)$. Find the moment generating function M where $M(t) = \mathbb{E}[e^{tX}]$ for all real t.

Problem 18. If the random variable X is normal with mean μ and variance σ^2 then $Y = \exp(X)$ is said to be log-normal. Compute the mean and variance of Y.

Problem 19. Let X be a random variable with density function f_X where $f_X(t) = \frac{1}{2} e^{-|t|}$ for all $t \in \mathbb{R}$. Find the characteristic function of X. Let Y be a Cauchy random variable. Find the characteristic function of Y. (You may need to evaluate a contour integral.)