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Example Sheet 1 - Michaelmas 2006

Problem 1. Show that if $\mathcal{F}$ is a sigma-field on a set Ω then that both Ω and $\emptyset$ are elements of $\mathcal{F}$.

Solution 1. If A is in $\mathcal{F}$, so is the complement A^c. Hence the union $A \cup A^c = \Omega$ is in $\mathcal{F}$, and so is its complement $\Omega^c = \emptyset$.

Problem 2. Let $\mathcal{F}$ and $\mathcal{G}$ be two sigma-fields on a set Ω. Prove that $\mathcal{F} \cap \mathcal{G}$ is also a sigma-field on Ω. Show by example that $\mathcal{F} \cup \mathcal{G}$ may fail to be a sigma-field.

Solution 2. If A is in $\mathcal{F} \cap \mathcal{G}$ then A is in both $\mathcal{F}$ and $\mathcal{G}$. Since $\mathcal{F}$ and $\mathcal{G}$ are both sigma-fields, the complement A^c is in both $\mathcal{F}$ and $\mathcal{G}$, and thus A^c is in the intersection $\mathcal{F} \cap \mathcal{G}$. Similarly, if $A_1, A_2, \ldots \in \mathcal{F} \cap \mathcal{G}$ then $A_1, A_2, \ldots \in \mathcal{F}$ and $A_1, A_2, \ldots \in \mathcal{G}$. The union $\cup_{i=1}^{\infty} A_i$ is in both $\mathcal{F}$ and $\mathcal{G}$, and thus is an element of $\mathcal{F} \cap \mathcal{G}$. This completes the verification that $\mathcal{F} \cap \mathcal{G}$ is a sigma-field.

[Note that the above argument can be extended to the intersection of an arbitrary collection of sigma-fields. That is, if $\{\mathcal{F}_i\}_{i \in I}$ is a (finite, countable, or even uncountable) collection of sigma-fields on Ω, then $\bigcap_{i \in I} \mathcal{F}_i$ is also a sigma-field. It is because of this observation that we can, for instance, define the Borel sigma-field as the smallest sigma-field containing the intervals.]

Now let $\Omega = \{1, 2, 3\}$, $\mathcal{F} = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$ and $\mathcal{G} = \{\emptyset, \{2\}, \{1, 3\}, \{1, 2, 3\}\}$. The union $\mathcal{F} \cup \mathcal{G}$ contains both $\{1\}$ and $\{2\}$, yet does not contain $\{1\} \cup \{2\} = \{1, 2\}$. Hence $\mathcal{F} \cup \mathcal{G}$ is not a sigma-field.

Problem 3. Let $A_1, A_2, A_3, \ldots$ be a sequence of events such that $A_1 \subset A_2 \subset A_3 \subset \ldots$, and let $A = \bigcup_{n=1}^{\infty} A_n$. Prove that

$$\mathbb{P}(A) = \lim_{n \uparrow \infty} \mathbb{P}(A_n).$$

Solution 3. Let $B_1 = A_1$ and $B_n = A_n \backslash A_{n-1}$. The disjoint events $B_1, B_2, B_3, \ldots$ are such that $A_n = \bigcup_{i=1}^{n} B_i$ and $A = \bigcup_{i=1}^{\infty} B_i$ by construction. The conclusion follows by the countable additivity of $\mathbb{P}$:

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(\bigcup_{i=1}^{\infty} B_i) \\
&= \sum_{i=1}^{\infty} \mathbb{P}(B_i) \\
&= \lim_{n \uparrow \infty} \sum_{i=1}^{n} \mathbb{P}(B_i) \\
&= \lim_{n \uparrow \infty} \mathbb{P}(\bigcup_{i=1}^{n} B_i) = \lim_{n \uparrow \infty} \mathbb{P}(A_n).
\end{aligned}$$

Problem 4. Let X be a real-valued random variable, and let F_X be its distribution function. Show that

$$\lim_{t\downarrow -\infty} F_X(t) = 0, \quad \lim_{t\uparrow +\infty} F_X(t) = 1, \quad \mathbb{P}(X = t) = F_X(t) - \lim_{s\uparrow t} F_X(s).$$

In particular, the distribution function of X is defective if and only if

$$\mathbb{P}(X = +\infty \text{ or } X = -\infty) > 0.$$

Solution 4. Let $t_1, t_2, \ldots$ be any unbounded increasing sequence of numbers and let $A_n = \{X \le t_n\}$. The sequence of events $A_1 \subset A_2 \subset A_3 \subset \ldots$ is such that $\bigcup_{n=1}^{\infty} A_n = \Omega$, where Ω is the sample space on which the random variable X is defined. By question 1 we have

$$\lim_{n\uparrow\infty} F(t_n) = \lim_{n\uparrow\infty} \mathbb{P}(A_n) = \mathbb{P}(\Omega) = 1.$$

Similarly, since $\bigcup_{n=1}^{\infty}\{X > -t_n\} = \Omega$ we have the equality

$$\lim_{n\uparrow\infty} F(-t_n) = 1 - \lim_{n\uparrow\infty} \mathbb{P}(X > -t_n) = 0.$$

Finally, let $\epsilon_1, \epsilon_2, \ldots$ be any sequence of positive numbers decreasing to zero so that $\{X < t\} = \bigcup_{n=1}^{\infty}\{X \le t - \epsilon_n\}$.

$$\mathbb{P}(X = t) = \mathbb{P}(X \le t) - \mathbb{P}(X < t) = F(t) - \lim_{n\uparrow\infty} F(t - \epsilon_n).$$

Problem 5. Let $F : \mathbb{R} \to [0,1]$ be increasing, right-continuous, with

$$\lim_{t\downarrow -\infty} F(t) = 0 \text{ and } \lim_{t\uparrow +\infty} F(t) = 1.$$

Show that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there is a random variable X such that F is the distribution function of X. (Hint: Let the sample space Ω be the unit interval $[0,1]$, the events $\mathcal{F}$ the Borel sigma-field, and $\mathbb{P}$ the uniform measure.)

Solution 5. Let $X : [0,1] \to \mathbb{R}$ be defined as

$$X(\omega) = \inf\{t \in \mathbb{R} : F(t) \ge \omega\}$$

for $\omega \in [0,1]$. [Note that if F is strictly increasing and continuous, then $X = F^{-1}$.]

Since $X(\omega) \le t$ if and only if $F(t) \ge \omega$, the distribution function of X is given by

$$\mathbb{P}(X \le t) = \mathbb{P}([0, F(t)]) = F(t)$$

as desired.

[This contruction can be used to generate random numbers with a given law, assuming your computer can generate uniformly distributed random variables.]

Problem 6. A box contains 20 balls, labelled with the numbers from 1 to 20. Three balls are drawn at random from the box. Find the probability that 10 is the smallest label of the three balls.

Solution 6. There are $\binom{20}{3}$ ways to choose 3 balls from 20. There are $\binom{10}{2}$ ways to choose 2 balls from the balls labelled 11 to 20, $\binom{1}{1}$ ways to choose 1 ball with the label 10, and $\binom{9}{0}$ ways to choose no balls from those labelled 1 to 9. The desired probability is then

$$\frac{\binom{10}{2}\binom{1}{1}\binom{9}{0}}{\binom{20}{3}} = \frac{\frac{10\cdot 9}{2\cdot 1}}{\frac{20\cdot 19\cdot 18}{3\cdot 2\cdot 1}} = \frac{3}{76}.$$

Problem 7. A coin is tossed n times with probability p of heads on each toss. Let E be the event that the first toss lands heads. Let F be the event that there are exactly k heads. For which pairs of natural numbers n and k are the events E and F independent?

Solution 7. $\mathbb{P}(E) = p$ and $\mathbb{P}(F) = \binom{n}{k}p^k(1-p)^{n-k}$. Finally $\mathbb{P}(E \cap F) = \mathbb{P}(F|E)\mathbb{P}(E) = \binom{n-1}{k-1}p^{k-1}(1-p)^{n-k} \cdot p$. Equating the formulas for $\mathbb{P}(E)\mathbb{P}(F) = \mathbb{P}(E \cap F)$ and simplifying yields

$$p = \frac{k}{n}.$$

Problem 8. Let $X_1, \dots, X_n$ be independent and identically distributed random variables with mean μ and variance σ^2. Find the mean of the random variables $\bar{X}$ and S^2 where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

If $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, prove that $\bar{X}$ and S^2 are independent.

Solution 8. By the linearity of expectation:

$$\mathbb{E}(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) = \mu.$$

Also for $i = 1, \dots, n$ we have $X_i - \bar{X} = \frac{n-1}{n}(X_i - \mu) - \sum_{j \neq i}\frac{1}{n}(X_j - \mu)$. Since the summands are independent and mean zero we have

$$\mathbb{E}[(X_i - \bar{X})^2] = \left(\frac{n-1}{n}\right)^2 \mathbb{E}[(X_i - \mu)^2] + \sum_{j \neq i}\frac{1}{n^2}\mathbb{E}[(X_j - \mu)^2] = \frac{n-1}{n}\sigma^2$$

and hence

$$\mathbb{E}(S^2) = \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{E}[(X_i - \bar{X})^2] = \sigma^2.$$

Now assume that $X_1, \dots, X_n \sim N(\mu, \sigma^2)$. Since

$$\mathrm{Cov}(\bar{X}, X_i - \bar{X}) = \mathrm{Cov}(\bar{X}, X_i) - \mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0$$

we can conclude that the random variables $\bar{X}$ and $X_i - \bar{X}$ are independent, because the random variables $\bar{X}$ and $X_i - \bar{X}$ are jointly normal for each $i = 1, \ldots, n$. Hence $\bar{X}$ and $\sum_{i=1}^n (X_i - \bar{X})^2$ are independent as desired.

Problem 9. A population contains n animals. Every day, an animal is captured at random. If that animal has not been captured before, it is tagged and released. Prove that the expected number of days needed to tag all the animals is $n \sum_{i=1}^n 1/i$.

Solution 9. Let T_i be the random variable corresponding to the number of days between tagging of the i-th and $(i+1)$-th animal. Note if i animals have already been tagged, the probability of capturing an untagged animal is $1 - i/n$. Hence, $\mathbb{P}(T_i = 1) = 1 - i/n$, and in general, $\mathbb{P}(T_i = k) = (1 - i/n)(i/n)^{k-1}$. That is, T_i is a geometric random variable for $i = 1, \ldots, n-1$ with parameter $1 - i/n$. (The random variable T_0 is such that $T_0(\omega) = 1$ identically.) In particular, $\mathbb{E}(T_i) = \frac{n}{n-i}$. Thus the expected number of days need to tag all the animals is

$$\mathbb{E} \sum_{i=0}^{n-1} T_i = \sum_{i=0}^{n-1} \frac{n}{n-i} = n \sum_{i=1}^{n} \frac{1}{i}.$$

Problem 10. Let X and Y be independent Poisson random variables with parameters λ and μ respectively. What is the conditional distribution of X given $X + Y = n$?

Solution 10. First we find that the distribution of the sum $X + Y$ is Poisson with parameter $\lambda + \mu$:

$$\begin{aligned}
\mathbb{P}(X + Y = n) &= \sum_{i=0}^{n} \mathbb{P}(X = i, Y = n - i) \\
&= \sum_{i=0}^{n} \mathbb{P}(X = i)\mathbb{P}(Y = n - i) \\
&= \sum_{i=0}^{n} e^{-\lambda} \frac{\lambda^i}{i!} e^{-\mu} \frac{\mu^{n-i}}{(n-i)!} \\
&= e^{-(\lambda+\mu)} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} \lambda^i \mu^{n-i} \\
&= e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^n}{n!}.
\end{aligned}$$

Thus the conditional distribution of X given $X + Y = n$ is given by

$$\begin{aligned}
\mathbb{P}(X = i | X + Y = n) &= \frac{\mathbb{P}(X = i, X + Y = n)}{P(X + Y = n)} \\
&= \frac{e^{-\lambda} \frac{\lambda^i}{i!} e^{-\mu} \frac{\mu^{n-i}}{(n-i)!}}{e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^n}{n!}} \\
&= \binom{n}{i} \left(\frac{\lambda}{\lambda + \mu} \right)^i \left(1 - \frac{\lambda}{\lambda + \mu} \right)^{n-i}
\end{aligned}$$

which is the binomial distribution with parameters n and $\frac{\lambda}{\lambda+\mu}$.

Problem 11. Let X and Y be independent geometric random variables with the same parameter p. What is the conditional distribution of X given $X+Y=n$?

Solution 11. For $n = 2, 3, \ldots$ we have

$$\begin{aligned}\mathbb{P}(X+Y=n) &= \sum_{i=1}^{n-1} \mathbb{P}(X=i)\mathbb{P}(Y=n-i) \\ &= \sum_{i=1}^{n-1} p(1-p)^{i-1}p(1-p)^{n-i-1} \\ &= (n-1)p^2(1-p)^{n-2}\end{aligned}$$

and hence for $i = 1, \ldots, n-1$ we have

$$\begin{aligned}\mathbb{P}(X=i|X+Y=n) &= \frac{\mathbb{P}(X=i, X+Y=n)}{P(X+Y=n)} \\ &= \frac{p(1-p)^{i-1}p(1-p)^{n-i-1}}{(n-1)p^2(1-p)^{n-2}} \\ &= \frac{1}{n-1}\end{aligned}$$

and the conditional distribution of X given $X+Y=n$ is uniform on the set $\{1, \ldots, n-1\}$.

Problem 12. Let X be a geometric random variable. Prove that $\mathbb{P}(X=n+m|X>m) = \mathbb{P}(X=n)$ for all $m, n = 1, 2, 3, \ldots$. Why does one say that geometric random variables are memoryless?

Solution 12. Since X is geometric, there is a p such that $0 < p < 1$ and $\mathbb{P}(X=k) = p(1-p)^{k-1}$ for $k = 1, 2, \ldots$. Hence $\mathbb{P}(X>m) = \sum_{k=m+1}^{\infty} p(1-p)^{k-1} = (1-p)^m$ and

$$\begin{aligned}\mathbb{P}(X=n+m|X>m) &= \frac{\mathbb{P}(X=n+m, X>m)}{\mathbb{P}(X>m)} \\ &= \frac{p(1-p)^{m+n-1}}{(1-p)^m} \\ &= p(1-p)^{n-1} = \mathbb{P}(X=n)\end{aligned}$$

Problem 13. Let X be an exponential random variable. Prove that $\mathbb{P}(X > t+s|X > s) = \mathbb{P}(X > t)$ for all $s, t \geq 0$. Why does one say that exponential random variables are memoryless?

Solution 13. Since X is exponentially distributed, there is a $\lambda > 0$ such that the density f of X is $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. Hence for all $s \geq 0$ we have

$\mathbb{P}(X > s) = \int_s^\infty \lambda e^{-\lambda x} dx = e^{-\lambda s}$ and hence

$$\begin{aligned}
\mathbb{P}(X > t+s | X > s) &= \frac{\mathbb{P}(X > t+s, X > s)}{\mathbb{P}(X > s)} \\
&= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \\
&= e^{-\lambda t} = \mathbb{P}(X > t)
\end{aligned}$$

Problem 14. Let $X_1, \ldots, X_n$ be uncorrelated random variables with mean 0 and variance 1. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. By using Chebyshev's inequality, find the smallest n such that

$$\mathbb{P}(|\bar{X}| > 2) \leq \frac{1}{100}.$$

Now redo this problem under the additional assumption that the random variables $X_1, \ldots, X_n$ are jointly normal.

Solution 14. The sample mean $\bar{X}$ has mean zero and variance $1/n$. By the Chebyshev's inequality

$$\mathbb{P}(|\bar{X}| > 2) < \frac{1}{4n}.$$

Hence $n = 25$ suffices.

If $X_1, \ldots, X_n$ are jointly normal, then the random variable $Z = \sqrt{n}\bar{X}$ is a standard normal. By the symmetry of the standard normal density we have

$$\begin{aligned}
\mathbb{P}(|\bar{X}| > 2) &= \mathbb{P}(|Z| > 2\sqrt{n}) \\
&= 2(1 - \mathbb{P}(Z \leq 2\sqrt{n})).
\end{aligned}$$

Since $\Phi(2.58) \approx 0.995$, the answer is the smallest integer solution to $2\sqrt{n} \geq 2.58$, which is $n = 2$.

Problem 15. Let X and Y be independent exponential random variables with parameters λ and μ respectively. Show that $U = \min\{X, Y\}$ is exponential with parameter $\lambda + \mu$. Show that the events $\{U \leq t\}$ and $\{X < Y\}$ are independent for all t.

Solution 15. $\mathbb{P}(U > t) = \mathbb{P}(\min\{X, Y\} > t) = \mathbb{P}(X > t, Y > t) = \mathbb{P}(X > t)\mathbb{P}(Y > t) = e^{-(\lambda+\mu)t}$. Hence $\mathbb{P}(U \leq t) = 1 - e^{-(\lambda+\mu)t}$. That is, the random variable U has the exponential distribution with parameter $\lambda + \mu$.

The desired probabilities can be calculated by integrating the joint density of the appropriate region:

$$\begin{aligned}
\mathbb{P}(X < Y) &= \int_{x=0}^\infty \int_{y=x}^\infty \lambda\mu e^{-\lambda x + \mu y} dx\ dy \\
&= \int_{x=0}^\infty \lambda\mu e^{-(\lambda+\mu)y} dy \\
&= \frac{\lambda}{\lambda + \mu}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(U \le t, X < Y) &= \int_{x=0}^{t}\int_{y=x}^{\infty} \lambda\mu e^{-\lambda x+\mu y} dx\ dy \\
&= \int_{x=0}^{t} \lambda\mu e^{-(\lambda+\mu)y} dy \\
&= \frac{\lambda}{\lambda+\mu}(1-e^{-(\lambda+\mu)t}) \\
&= \mathbb{P}(X < Y)\mathbb{P}(U \le t)
\end{aligned}$$

and so the events $\{U \le t\}$ and $\{X < Y\}$ are independent.

Problem 16. Let X and Y be identically distributed random variables with finite mean and variance. Show that $U = X + Y$ and $V = X - Y$ are uncorrelated. Show by example that U and V need not be independent.

Solution 16. $\mathbb{E}(V) = \mathbb{E}(X-Y) = \mu-\mu = 0$ and $\mathbb{E}(UV) = \mathbb{E}(X^2-Y^2) = \sigma^2+\mu^2-(\sigma^2+\mu^2) = 0$

$$\mathrm{Cov}(U, V) = \mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V) = 0$$

Let X and Y be independent Bernoulli random variables with parameter p. Then the events $\{U = 2\} = \{X = 1, Y = 1\}$ and $\{V = 1\} = \{X = 1, Y = 0\}$ are not independent since

$$\mathbb{P}(U = 2, V = 1) = 0 \neq p^3(1-p) = \mathbb{P}(U = 2)\mathbb{P}(V = 1).$$

The random variables U and V are dependent.

Problem 17. Let $X \sim N(\mu, \sigma^2)$. Find the moment generating function M where $M(t) = \mathbb{E}(e^{tX})$ for all real t.

Solution 17. Since $X = \mu + \sigma Z$, where $Z \sim N(0, 1)$, we first compute

$$\begin{aligned}
\mathbb{E}(e^{tZ}) &= \int_{-\infty}^{\infty} e^{tz}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-(z-t)^2/2+t^2/2}dz \\
&= e^{t^2/2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\
&= e^{t^2/2}.
\end{aligned}$$

Now

$$\begin{aligned}
\mathbb{E}(e^{tX}) &= \mathbb{E}(e^{t(\mu+\sigma Z)}) \\
&= e^{t\mu}\mathbb{E}(e^{t\sigma Z}) \\
&= e^{t\mu+t^2\sigma^2/2}
\end{aligned}$$

Problem 18. If the random variable X is normal with mean μ and variance σ^2 then $Y = \exp(X)$ is said to be log-normal. Compute the mean and variance of Y.

Solution 18. As before, let $X = \mu + \sigma Z$ where $Z \sim N(0,1)$.

$$\begin{aligned}\mathbb{E}(Y) = \mathbb{E}(e^X) &= \mathbb{E}(e^{\mu+\sigma Z}) \\ &= e^{\mu+\sigma^2/2}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}(Y^2) = \mathbb{E}(e^{2X}) &= \mathbb{E}(e^{2\mu+2\sigma Z}) \\ &= e^{2\mu+2\sigma^2}\end{aligned}$$

so that

$$\mathrm{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$

Problem 19. Let X be a random variable with density function f_X where $f_X(t) = \frac{1}{2}e^{-|t|}$ for all $t \in \mathbb{R}$. Find the characteristic function of X. Let Y be a Cauchy random variable. Find the characteristic function of Y. (You may need to evaluate a contour integral.)

Solution 19.

$$\phi_X(s) = \int_{-\infty}^{\infty} \frac{1}{2}e^{-|t|}e^{ist}\,dt = \frac{1}{1+s^2}$$

To compute the characteristic function of Y, first suppose $s > 0$.

$$\begin{aligned}\phi_Y(s) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{ist}}{1+t^2}\,dt \\ &= \frac{1}{\pi}\lim_{R\uparrow\infty}\int_{-R}^{R} \frac{e^{ist}}{1+t^2}\,dt \\ &= \frac{1}{\pi}\lim_{R\uparrow\infty}\left(\oint_{\Gamma_R} \frac{e^{isz}}{1+z^2}\,dz - \int_0^{\pi} \frac{e^{isRe^{i\theta}+i\theta}R}{1+R^2e^{2i\theta}}d\theta\right)\end{aligned}$$

where Γ_R is the contour in the complex plane $\{t + 0i : t \in [-R, R]\} \cup \{Re^{i\theta} : \theta \in [0, \pi]\}$ where the integration is performed counterclockwise on Γ_R. Since $z \mapsto \frac{e^{isz}}{1+z^2}$ in meromorphic in the interior of Γ_R with simple pole at i we have that

$$\oint_{\Gamma_R} \frac{e^{isz}}{1+z^2}\,dz = \pi e^{-s}$$

by the residue theorem. Also

$$\left|\int_0^{\pi} \frac{e^{isRe^{i\theta}+i\theta}R}{1+R^2e^{2i\theta}}d\theta\right| \le \int_0^{\pi} \frac{e^{-sR\sin(\theta)}R}{R^2-1}d\theta \to 0$$

since $s \geq 0$, so that

$$\phi_Y(s) = e^{-s} \text{ for } s \geq 0.$$

A similar computation for the case $s < 0$ using the contour $\Gamma_R^- = \{t + 0i : t \in [-R, R]\} \cup \{Re^{i\theta} : \theta \in [\pi, 2\pi]\}$ shows that

$$\phi_Y(s) = e^{-|s|} \text{ for all } s \in \mathbb{R}.$$

[The point of this exercise is to see a special case of the Fourier inversion formula: If X is a random variable with differentiable density f_X and characteristic function ϕ_X then

$$f_X(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist} \phi_X(s)\, ds.$$

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