SYMMETRIC MARTINGALES AND SYMMETRIC SMILES

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Abstract. A local martingale X is called arithmetically symmetric if the conditional distribution of $X_T - X_t$ is symmetric given $\mathcal{F}_t$, for all $0 \le t \le T$. Letting $\mathcal{F}_t^T = \mathcal{F}_t \vee \sigma(\langle X \rangle_T)$, the main result of this note is that for a continuous local martingale X the following are equivalent

(1) X is arithmetically symmetric.
(2) The conditional distribution of X_T given $\mathcal{F}_t^T$ is $N(X_t, \langle X \rangle_T - \langle X \rangle_t)$ for all $0 \le t \le T$.
(3) X is a local martingale for the enlarged filtration $(\mathcal{F}_t^T)_{t \ge 0}$ for each $T \ge 0$.

The notion of a geometrically symmetric martingale is also defined and characterized as the Doléans-Dade exponential of an arithmetically symmetric local martingale. As an application of these results, we show that a market model of the implied volatility surface that is initially flat and that remains symmetric for all future times must be the Black–Scholes model.

1. Introduction

Let $X = (X_t)_{t \ge 0}$ be a real-valued continuous local martingale for a filtration $(\mathcal{F}_t)_{t \ge 0}$. We say that X is arithmetically symmetric if the conditional distribution of $X_T - X_t$ is symmetric given $\mathcal{F}_t$ for all $0 \le t \le T$. More precisely, X is arithmetically symmetric if

$$\mathbb{E}[f(X_T - X_t)|\mathcal{F}_t] = \mathbb{E}[f(X_t - X_T)|\mathcal{F}_t] \tag{1}$$

almost surely, for all $0 \le t \le T$ and bounded measurable f. The main result of this note is Theorem 2.2 which says that X is arithmetically symmetric if and only if for all $0 \le t \le T$ the conditional distribution of the increment $X_T - X_t$ given the increment $\langle X \rangle_T - \langle X \rangle_t$ of quadratic variation is normal with mean zero and variance $\langle X \rangle_T - \langle X \rangle_t$, independent of $\mathcal{F}_t$. An easy corollary of the main result is that if X is arithmetically symmetric and if the marginal distribution of X_t is normal with mean 0 and variance t for all $t \ge 0$, then X is a standard Brownian motion.

Ocone [12] studied the related problem of characterizing local martingales with conditionally independent increments. He showed that if the continuous local martingales X and $\int (\mathbb{1}_{[0,s]} - \mathbb{1}_{(s,\infty)})dX$ have the same law for each $s \ge 0$, then X has the form $X_t = X_0 + W_{A_t}$ for a standard Brownian motion W and an independent non-decreasing process A. For this reason, such local martingales are often called Ocone martingales. Note that the condition of arithmetic symmetry is weaker than Ocone's condition, since his condition directly implies (if the filtration is generated by X) the almost sure equality of the conditional expectations

$$\mathbb{E}[f(X_{t_1} - X_s, \dots, X_{t_n} - X_s)|\mathcal{F}_s] = \mathbb{E}[f(X_s - X_{t_1}, \dots, X_s - X_{t_n})|\mathcal{F}_s]$$

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for all $0 \leq s \leq t_1 \leq \ldots \leq t_n$ and bounded measurable f. Nevertheless, Theorem 2.2 suggests a natural question: If a continuous local martingale X is arithmetically symmetric, is X Ocone? Unfortunately, we do not resolve this question here. See the paper of Dubins, Emery, and Yor [5] for a discussion of the connection between Ocone martingales and a conjecture on the ergodicity of the Lévy transform. Further invariance properties of Ocone martingales with respect to Girsanov's theorem and to the reflection principle can be found in the papers of Vostrikova and Yor [18] and Chaumont and Vostrikova [4] respectively.

A related notion of symmetry is defined similarly: we say that a positive local martingale S is geometrically symmetric if

$$\mathbb{E}\left[g\left(\frac{S_T}{S_t}\right)|\mathcal{F}_t\right] = \mathbb{E}\left[\frac{S_T}{S_t}g\left(\frac{S_t}{S_T}\right)|\mathcal{F}_t\right]. \tag{2}$$

almost surely, for all $0 \leq t \leq T$ and bounded measurable g. If we let $g(y) = 1$ for all $y > 0$, then we see that geometric symmetry implies that S is a true martingale. More interestingly, if S is a positive continuous martingale, then we can define a local martingale X by $X_t = \log S_t + \langle \log S\rangle_t/2$ or equivalently $\log S_t = X_t - \langle X\rangle_t/2$. In Theorem 3.1 we show that S is geometrically symmetric if and only if X is arithmetically symmetric.

As indicated by Bates [1], Schroder [15], and others, the financial motivation for studying geometrically symmetric martingales is the observation that S is geometrically symmetric if and only if the put-call symmetry formula

$$\mathbb{E}\left[(S_T - K)^+|\mathcal{F}_t\right] = \frac{K}{S_t}\mathbb{E}\left[\left(\frac{S_t^2}{K} - S_T\right)^+|\mathcal{F}_t\right]$$

holds almost surely for all $0 \leq t \leq T$ and strike $K > 0$. That is, if the price of a stock is modelled as the geometrically symmetric martingale S for a risk-neutral measure $\mathbb{P}$, then there is no arbitrage in the market (assuming for simplicity zero interest and dividend rates) if the time-t price of a call option struck at K is equal to K/S_t times the time-t price of a put option struck at S_t^2/K.

Renault and Touzi [13] showed that if S comes from the stochastic volatililty model

$$\begin{aligned} dS_t &= S_t\sqrt{V_t}dW_t^1 \\ dV_t &= \alpha(t, V_t)dt + \beta(t, V_t)dW_t^2 \end{aligned}$$

where W^1 and W^2 are independent Brownian motions, then S is geometrically symmetric. Carr and Lee [3] proved a converse of this result: Suppose S and V are given as above, and the Brownian motions W^1 and W^2 have correlation ρ. Then S satisfies equation (2) for $t = 0$ and all $T \geq 0$ and bounded measurable g if and only if $\rho = 0$. Theorem 3.1 of this paper is in the spirit of the Carr–Lee result: if we replace the special structure of a stochastic volatility model with the assumption that equation (2) holds for all $0 \leq t \leq T$, then Theorem 3.1 says that the distributon of $\log S_T$ is simply a mixture of normal distributions for all T.

Carr and Lee showed that one way to interpret the geometric symmetry condition is via the Black–Scholes implied volatility. Indeed, since the Black–Scholes model also satisfies put-call symmetry, we see that S is geometrically symmetric if and only if its implied volatility is symmetric in the sense that

$$\Sigma_t(\tau, m) = \Sigma_t(\tau, 1/m)$$

almost surely for all $t \geq 0, \tau > 0, m > 0$, where the implied volatility is defined as the unique (up to null sets) non-negative solution to the equation

$$\mathbb{E}\left[\left(\frac{S_{t+\tau}}{S_t} - m\right)^+ \big| \mathcal{F}_t\right] = \mathrm{BS}(\tau \Sigma_t(\tau, m)^2, m),$$

the Black–Scholes call price function BS is given by

$$\mathrm{BS}(v, m) = \begin{cases} \Phi\big(-\frac{\log m}{\sqrt{v}} + \frac{\sqrt{v}}{2}\big) - m\Phi\big(-\frac{\log m}{\sqrt{v}} - \frac{\sqrt{v}}{2}\big) & \text{if } v > 0 \\ (1-m)^+ & \text{if } v = 0 \end{cases} \tag{3}$$

and Φ is the distribution function of a standard normal random variable. In particular, we find in Theorem 4.1 that if the initial implied volatility surface is flat, that is if $\Sigma_0(\tau, m) = \sigma_0$ for all $\tau > 0, m > 0$, and the arbitrage-free dynamics of the random field Σ are constrained to be symmetric almost surely for all future times, then the surface satisfies $\Sigma_t(\tau, m) = \sigma_0$ almost surely for all $t \geq 0, \tau > 0, m > 0$.

The paper proceeds as follows: In Section 2 the main theorem is stated and proven, characterizing continuous local martingales with arithmetic symmetry. In Section 3, the case of geometric symmetry is studied by similar methods. In Section 4, we show that a market model of implied volatility that begins flat and remains symmetric must be the Black–Scholes model. In Section 5, we conclude.

2. The arithmetically symmetric case

Let X be a real-valued local martingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$. We make the following assumption throughout:

Assumption 2.1. *The filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions, with $\mathcal{F}_0$ trivial. Furthermore, every $(\mathcal{F}_t)_{t\geq 0}$-martingale is continuous.*

Note that this assumption is satisfied if the filtration is the augmentation of the filtration generated by a (possibly multi-dimensional) Brownian motion. Filtrations of this kind occur frequently in local and stochastic volatility models, so Assumption 2.1 is not onerous for the financial applications motivating this study.

We will let $\mathcal{F}_t^T = \mathcal{F}_t \vee \sigma(\langle X \rangle_T)$ be the smallest sigma-field containing $\mathcal{F}_t$ for which the random variable $\langle X \rangle_T$ is measurable. Note that since $\langle X \rangle_T$ is $\mathcal{F}_T$-measurable, we have $\mathcal{F}_u = \mathcal{F}_u^T$ for $u \geq T$. We now come to the main theorem.

Theorem 2.2. *The following are equivalent*

1. *X is arithmetically symmetric.*
2. *The conditional distribution of X_T given $\mathcal{F}_t^T$ is $N(X_t, \langle X \rangle_T - \langle X \rangle_t)$ for all $0 \leq t \leq T$.*
3. *X is a local martingale for the enlarged filtration $(\mathcal{F}_t^T)_{t\geq 0}$ for all $T \geq 0$.*

Proof. (2) $\Rightarrow$ (1) By the assumption of conditional normality we have the computation

$$\begin{aligned}\mathbb{E}[e^{i\theta(X_T - X_t)}|\mathcal{F}_t] &= \mathbb{E}\{\mathbb{E}[e^{i\theta(X_T - X_t)}|\mathcal{F}_t^T]|\mathcal{F}_t\} \\ &= \mathbb{E}[e^{-\theta^2(\langle X \rangle_T - \langle X \rangle_t)/2}|\mathcal{F}_t]\end{aligned}$$

for all $\theta \in \mathbb{R}$. Since the conditional characteristic function of the increment is even and characterizes the conditional distribution, we are done.

(1) ⇒ (2) This is the main part of the proof. Without loss of generality, assume henceforth that $X_0 = 0$ and fix non-zero $\theta \in \mathbb{R}$ and $T > 0$. Let $M = (M_t)_{t\in[0,T]}$ be the bounded, complex-valued martingale

$$M_t = \mathbb{E}[e^{i\theta X_T}|\mathcal{F}_t].$$

Note that M is continuous by Assumption 2.1.

Now, the process $(e^{-2i\theta X_t}M_t)_{t\in[0,T]}$ is also a martingale since by the assumption of symmetry, we have the equalities

$$\begin{aligned} M_t &= e^{i\theta X_t}\mathbb{E}[e^{i\theta(X_T-X_t)}|\mathcal{F}_t] \\ &= e^{i\theta X_t}\mathbb{E}[e^{i\theta(X_t-X_T)}|\mathcal{F}_t] \\ &= e^{2i\theta X_t}\mathbb{E}[e^{-i\theta X_T}|\mathcal{F}_t] \end{aligned}$$

But by Itô's formula, we have

$$d(e^{-2i\theta X_t}M_t) = e^{-2i\theta X_t}[dM_t - 2i\theta M_t e^{-2i\theta X_t}dX_t] + 2i\theta e^{-2i\theta X_t}[i\theta M_t d\langle X\rangle_t - d\langle M, X\rangle_t]$$

and hence

$$d\langle M, X\rangle_t = i\theta M_t d\langle X\rangle_t. \tag{4}$$

Now, fix a real ϕ such that $2\phi\theta \geq \phi^2$ and define a complex process Z by

$$Z_t = M_t e^{-i\phi X_t - (2\phi\theta-\phi^2)\langle X\rangle_t/2}.$$

By Itô's formula and equation (4) we see

$$dZ_t = e^{-i\phi X_t - (2\phi\theta-\phi^2)\langle X\rangle_t/2}[dM_t - i\phi M_t dX_t]$$

and hence Z is a local martingale. But since $|Z_t| \leq 1$ for all $t \in [0, T]$, the local martingale Z is in fact a true martingale implying

$$\begin{aligned} \mathbb{E}[e^{i\theta X_T}|\mathcal{F}_t]e^{-i\phi X_t-(2\phi\theta-\phi^2)\langle X\rangle_t/2} &= Z_t \\ &= \mathbb{E}[Z_T|\mathcal{F}_t] \\ &= \mathbb{E}[M_T e^{-i\phi X_T-(2\phi\theta-\phi^2)\langle X\rangle_T/2}|\mathcal{F}_t] \\ &= \mathbb{E}[e^{i(\theta-\phi)X_T-(2\phi\theta-\phi^2)\langle X\rangle_T/2}|\mathcal{F}_t] \end{aligned}$$

where we have used the terminal condition $M_T = e^{i\theta X_T}$. Rearranging the above equalities yields

$$\begin{aligned} \mathbb{E}[e^{i\theta(X_T-X_t)}|\mathcal{F}_t] &= \mathbb{E}[e^{i(\theta-\phi)(X_T-X_t)-(2\phi\theta-\phi^2)(\langle X\rangle_T-\langle X\rangle_t)/2}|\mathcal{F}_t] \\ &= \mathbb{E}[e^{-\theta^2(\langle X\rangle_T-\langle X\rangle_t)/2}|\mathcal{F}_t] \end{aligned}$$

where the last equality is the case $\phi = \theta$. Letting $p = \theta - \phi$ and $q^2 = 2\phi\theta - \phi^2$ we have

$$\mathbb{E}[e^{ip(X_T-X_t)-q^2(\langle X\rangle_T-\langle X\rangle_t)/2}|\mathcal{F}_t] = \mathbb{E}[e^{-(p^2+q^2)(\langle X\rangle_T-\langle X\rangle_t)/2}|\mathcal{F}_t]$$

for all $p, q \in \mathbb{R}$ and all $0 \leq t \leq T$. The above equation implies

$$\mathbb{E}[e^{ip(X_T-X_t)}h(\langle X\rangle_T)|\mathcal{F}_t] = \mathbb{E}[e^{-p^2(X_T-X_t)/2}h(\langle X\rangle_T)|\mathcal{F}_t]$$

for all continuous and bounded h. We therefore have

$$\mathbb{E}[e^{ip(X_T-X_t)}|\mathcal{F}_t \vee \sigma(\langle X\rangle_T)] = e^{-p^2(\langle X\rangle_T-\langle X\rangle_t)/2}$$

for all $p \in \mathbb{R}$, proving the claim.

(2) ⇒ (3) Fix real $p < q$ and define a local martingale N by

$$N_t = e^{ipX_t + p^2 \langle X \rangle_t / 2}$$

and a (continuous) martingale Y by

$$Y_t = \mathbb{E}[e^{-q^2 \langle X \rangle_T / 2} | \mathcal{F}_t].$$

Note the equality

$$\begin{aligned} \mathbb{E}[N_T Y_T | \mathcal{F}_t] &= \mathbb{E}[e^{ipX_T + (p^2 - q^2) \langle X \rangle_T / 2} | \mathcal{F}_t] \\ &= e^{ipX_t + p^2 \langle X \rangle_t / 2} \mathbb{E}[e^{-q^2 \langle X \rangle_T / 2} | \mathcal{F}_t] \\ &= N_t Y_t. \end{aligned}$$

Since NY is a martingale, we must have $\langle N, Y \rangle = 0$ which implies $\langle X, Y \rangle = 0$. Define a sequence of $(\mathcal{F}_t)_{t \geq 0}$ stopping times by $\tau_n = \inf\{t \geq 0 : |X_t| \geq n\}$ and the stopped process X^n by $X^n_t = X_{t \wedge \tau_n}$. Since the process $X^n Y$ is a bounded martingale, we have for all $0 \leq s \leq t \leq T$ the calculation

$$\begin{aligned} \mathbb{E}[X_{t \wedge \tau_n} e^{-q^2 \langle X \rangle_T / 2} | \mathcal{F}_s] &= \mathbb{E}[X_{t \wedge \tau_n} Y_t | \mathcal{F}_s] \\ &= X_{s \wedge \tau_n} Y_s \\ &= X_{s \wedge \tau_n} \mathbb{E}[e^{-q^2 \langle X \rangle_T / 2} | \mathcal{F}_s] \end{aligned}$$

for each $n > 0$. Since

$$\mathbb{E}[X^n_t | \mathcal{F}^T_s] = X^n_s,$$

for each n, the process X is a local martingale for the filtration $(\mathcal{F}^T_t)_{t \geq 0}$.

(3) ⇒ (2) Suppose X is a local martingale for the filtration $(\mathcal{F}^T_t)_{t \geq 0}$ for all $T \geq 0$. As before, fix $p \in \mathbb{R}$ and let

$$N_t = e^{ipX_t + p^2 \langle X \rangle_t / 2}.$$

Define a sequence of $(\mathcal{F}_t)_{t \geq 0}$-stopping times by $\sigma_n = \inf\{t \geq 0 : |X_t| \vee \langle X \rangle_t \geq n\}$ and the stopped processes $X^n_t = X_{t \wedge \sigma_n}$ and $N^n_t = N_{t \wedge \sigma_n}$. First note that X^n is a $(\mathcal{F}^T_t)_{t \geq 0}$-martingale since X^n is a bounded local martingale. Since

$$N^n_t = 1 + \int_0^t N^n_s dX^n_s$$

and

$$\mathbb{E}\left[\int_0^T |N^n_s|^2 \langle X^n \rangle_s\right] \leq e^{p^2 n/2} n < \infty.$$

we see that N^n is also a $(\mathcal{F}^T_t)_{t \geq 0}$-martingale.

Now fix $q > p$. Since $|N_T^n e^{-q^2\langle X\rangle_T/2}| \le 1$ for all n, the dominated convergence theorem implies

$$\begin{aligned}
\mathbb{E}[e^{ipX_T+(p^2-q^2)\langle X\rangle_T/2}|\mathcal{F}_t] &= \mathbb{E}[\lim_{n\uparrow\infty} N_T^n e^{-q^2\langle X\rangle_T/2}|\mathcal{F}_t] \\
&= \lim_{n\uparrow\infty} \mathbb{E}[N_T^n e^{-q^2\langle X\rangle_T/2}|\mathcal{F}_t] \\
&= \lim_{n\uparrow\infty} \mathbb{E}[\mathbb{E}\{N_T^n e^{-q^2\langle X\rangle_T/2}|\mathcal{F}_t^T\}|\mathcal{F}_t] \\
&= \lim_{n\uparrow\infty} N_t^n \mathbb{E}[e^{-q^2\langle X\rangle_T/2}|\mathcal{F}_t] \\
&= e^{ipX_t+p^2\langle X\rangle_t/2}\mathbb{E}[e^{-q^2\langle X\rangle_T/2}|\mathcal{F}_t]
\end{aligned}$$

where we have used the facts that the random variable $e^{-q^2\langle X\rangle_T/2}$ is $\mathcal{F}_t^T$-measurable and that $N_t^n = \mathbb{E}[N_T^n|\mathcal{F}_t^T]$ is $\mathcal{F}_t$-measurable. The above identity establishes the claim. ☐

Remark 1. Note if X satisfies condition (2) of Theorem 2.2 then

$$\mathbb{E}[e^{i\theta(X_T-X_t)}|\langle X\rangle_T - \langle X\rangle_t] = e^{-\theta^2(\langle X\rangle_T-\langle X\rangle_t)/2}$$

so that the conditional distribution of the increment $X_T - X_t$ given the increment $\langle X\rangle_T - \langle X\rangle_t$ of quadratic variation is normal, and independent of $\mathcal{F}_t$. However, if X were Ocone, that is of the form $X_t = X_0 + W_{A_t}$ for a Brownian motion W and independent non-decreasing process A, then the conditional distribution $X_T - X_t$ would be normally distributed given the much larger sigma-field $\sigma(\langle X\rangle_u - \langle X\rangle_t, u \ge t)$.

A corollary of this theorem is a characterization of Brownian motion as the arithmetically symmetric local martingale starting at zero with two absolute moments agreeing with those of Brownian motion. For instance, if $\mathbb{E}[X_t^4] = 3\mathbb{E}[X_t^2]^2 = 3t^2$ for all $t \ge 0$, then X is a standard Brownian motion.

Corollary 2.3. *Suppose X is arithmetically symmetric with $X_0 = 0$, and that there exists $0 < m < n$ such that $\mathbb{E}[|X_t|^m] = t^{m/2}C_m$ and $\mathbb{E}[|X_t|^n] = t^{n/2}C_n$ for all $t > 0$, where $C_k = \pi^{-1/2}2^{k/2}\Gamma(k/2 + 1/2)$ and $\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du$. Then X is a standard Brownian motion.*

Proof. By Theorem 2.2, we know the conditional law of X_t given $\langle X\rangle_t$ is normal. But by assumption

$$\begin{aligned}
\mathbb{E}[\langle X\rangle_t^{n/2}] &= \mathbb{E}[|X_t|^n]/C_n \\
&= t^{n/2} \\
&= \mathbb{E}[\langle X\rangle_t^{m/2}]^{n/m}.
\end{aligned}$$

But Jensen's inequality, with the strictly convex function $h(x) = x^{n/m}$, holds with equality only for constants. Hence $\langle X\rangle_t = t$ almost surely for all $t \ge 0$. The claim follows from Lévy's characterization of Brownian motion. ☐

Remark 2. It is clear from the proof that one could replace the moment condition by, for instance, the condition that $\mathbb{E}[e^{i\lambda X_t}] = e^{-\lambda^2 t/2}$ and $\mathbb{E}[e^{i\mu X_t}] = e^{-\mu^2 t/2}$ for all $t \ge 0$, for some real λ and μ with $|\lambda| \ne |\mu|$. In particular, if X is arithmetically symmetric and if the unconditional distribution of X_t is normal with mean 0 and variance t for all $t \ge 0$, then X is a standard Brownian motion.

The above comments suggest the open following problem: If X is a continuous local martingale such that $X_t \sim N(0,t)$ for all $t \geq 0$, does it follow that X is a Brownian motion? The answer is no if X were not assumed continuous. See Madan and Yor [11] and Hamza and Klebaner [8] for examples. On the other hand, if X is a square-integrable continuous martingale with $\langle X \rangle_t = \int_0^t \alpha_s^2 ds$ for a bounded, continuous, predictable process α, then $\mathbb{E}[\alpha_t^2 | X_t] = 1$ almost surely for all $t \geq 0$. In particular, if α is of the form $\alpha_t = a(t, X_t)$ for a deterministic function a, then X must be a Brownian motion by Lévy's characterization. This claim can be proven by fixing $\lambda \in \mathbb{R}$. By Itô's formula,

$$e^{i\lambda X_t} = 1 + i\lambda \int_0^t e^{i\lambda X_s} dX_s - \frac{\lambda^2}{2} \int_0^t e^{i\lambda X_s} \alpha_s^2 ds$$

Since the integrand is bounded, the stochastic integral $\int_0^t e^{i\lambda X_s} dX_s$ defines a martingale so that by Fubini's theorem we have

$$e^{-\lambda^2 t/2} = 1 - \frac{\lambda^2}{2} \int_0^t \mathbb{E}[e^{i\lambda X_s} \alpha_s^2] ds.$$

By the assumed continuity and boundedness of α we may differentiate both sides with respect t to conclude

$$e^{-\lambda^2 t/2} = \mathbb{E}[e^{i\lambda X_t} \alpha_t^2]$$

and hence

$$\mathbb{E}[e^{i\lambda X_t}(\alpha_t^2 - 1)] = 0$$

for all λ. The claim now follows.

3. The geometrically symmetric case

We now suppose that $S = (S_t)_{t \geq 0}$ is a strictly positive continuous martingale. The main theorem of this section is the following characterization:

Theorem 3.1. *The martingale S is geometrically symmetric if and only if S is of the form*

$$S_t = S_0 e^{X_t - \langle X \rangle_t / 2}$$

for a arithmetically symmetric local martingale X.

Before we begin, we need a small lemma.

Lemma 3.2. *The martingale S is geometrically symmetric if and only if*

$$\mathbb{E}\left[\left(\frac{S_T}{S_t} \right)^p | \mathcal{F}_t \right] = \mathbb{E}\left[\left(\frac{S_T}{S_t} \right)^{1-p} | \mathcal{F}_t \right]$$

for all complex $p = a + bi$ with $a \in [0,1]$ and all $0 \leq t \leq T$.

Proof of Lemma 3.2. Note

$$\left| \mathbb{E}\left[\left(\frac{S_T}{S_t} \right)^p | \mathcal{F}_t \right] \right| \leq \mathbb{E}\left[\left(\frac{S_T}{S_t} \right)^a | \mathcal{F}_t \right] \leq \mathbb{E}\left[\frac{S_T}{S_t} | \mathcal{F}_t \right]^a = 1$$

by Hölder's inequality, so that the conditional moments appearing in the lemma are finite almost surely.

The 'only if' direction is proven in [3]: Just take $g(x) = (x \wedge n)^p$ in the definition of symmetry, let $n \to \infty$, and apply the conditional dominated convergence theorem.

The 'if' direction is proven as follows. Fix $T > 0$. Given a martingale S satisfying the moment condition, we can construct another martingale by defining a measure $\hat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ with the density process $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = \frac{S_T}{S_0}$, with corresponding expectation operator $\hat{\mathbb{E}}$, and let $\hat{S}_t = \frac{S_0^2}{S_t}$. Now note

$$\begin{aligned}\hat{\mathbb{E}}\left[\left(\frac{\hat{S}_T}{\hat{S}_t}\right)^p |\mathcal{F}_t\right] &= \mathbb{E}\left[\left(\frac{S_T}{S_t}\right)^{1-p} |\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\left(\frac{S_T}{S_t}\right)^{p} |\mathcal{F}_t\right]\end{aligned}$$

for all $p \in (0,1)$ and $0 \le t \le T$. That is to say, the conditional moment generating function of $\log(S_T/S_t)$ under the measure $\mathbb{P}$ agrees on the interval $(0,1)$ with the the conditional moment generating function of $\log(\hat{S_T}/\hat{S}_t)$ under the measure $\hat{\mathbb{P}}$. Since knowledge of a moment generating function on an open interval characterizes the distribution, we have

$$\begin{aligned}\mathbb{E}\left[g\left(\frac{S_T}{S_t}\right) |\mathcal{F}_t\right] &= \hat{\mathbb{E}}\left[g\left(\frac{\hat{S}_T}{\hat{S}_t}\right) |\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\frac{S_T}{S_t} g\left(\frac{S_t}{S_T}\right) |\mathcal{F}_t\right].\end{aligned}$$

for all bounded measurable g, and we are done. $\square$

Proof of Theorem 3.1. The 'if' direction is an application of Lemma 3.2. Indeed, if $S_t = S_0 e^{X_t/2 - \langle X \rangle_t/2}$ with X being arithmetically symmetric, the conditional expectation

$$\mathbb{E}\left[\left(\frac{S_T}{S_t}\right)^p |\mathcal{F}_t\right] = \mathbb{E}\left[e^{-\frac{1}{2}(1-p)p(\langle X \rangle_T - \langle X \rangle_t)} |\mathcal{F}_t\right]$$

can be computed by first conditioning on $\mathcal{F}_t^T = \mathcal{F}_t \vee \sigma(\langle X \rangle_T)$ and applying the conditional normality. Since the right hand side of the above equation is unchanged if p is replaced with $1-p$, we are done.

As in the proof of Theorem 2.2 the 'only if' direction is more involved, but the same ideas can be made to work. Fix $T > 0$, and suppose that S is geometrically symmetric. Define a measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{S_T^{1/2}}{\mathbb{E}[S_T^{1/2}]}$$

with corresponding expectation $\tilde{\mathbb{E}}$. Letting $Y = \log S$, we have by the Lemma 3.2 the equality

$$\begin{aligned}\tilde{\mathbb{E}}[e^{i\theta(Y_T - Y_t)}|\mathcal{F}_t] &= \frac{\mathbb{E}[e^{(1/2+i\theta)(Y_T - Y_t)}|\mathcal{F}_t]}{\mathbb{E}[e^{Y_T/2}|\mathcal{F}_t]} \\ &= \frac{\mathbb{E}[e^{(1/2-i\theta)(Y_T - Y_t)}|\mathcal{F}_t]}{\mathbb{E}[e^{Y_T/2}|\mathcal{F}_t]} \\ &= \tilde{\mathbb{E}}[e^{-i\theta(Y_T - Y_t)}|\mathcal{F}_t]\end{aligned}$$

for all $t \in [0,T]$ and real θ.

Following the proof of Theorem 2.2, we find that

$$\tilde{\mathbb{E}}[e^{ip(Y_T-Y_t)-q^2(\langle Y\rangle_T-\langle Y\rangle_t)/2}|\mathcal{F}_t] = \tilde{\mathbb{E}}[e^{-(p^2+q^2)(\langle Y\rangle_T-\langle Y\rangle_t)/2}|\mathcal{F}_t]$$

for all real p, q for all $t \in [0, T]$. Unlike before, however, at this stage we are not finished, as the above expression involve the measure $\tilde{\mathbb{P}}$, rather than the original measure $\mathbb{P}$. We conclude by noting that the moment generating function

$$r \mapsto \tilde{\mathbb{E}}[e^{r(Y_T-Y_t)-q^2(\langle Y\rangle_T-\langle Y\rangle_t)/2}|\mathcal{F}_t] \le \frac{\mathbb{E}[e^{(1/2+r)(Y_T-Y_t)}|\mathcal{F}_t]}{\mathbb{E}[e^{Y_T/2}|\mathcal{F}_t]}$$

is finite almost surely for $r \in [-1/2, 1/2]$ and hence can be analytically continued to the strip $\{r = a + bi : a \in (-1/2, 1/2)\}$. Therefore, we have

$$\tilde{\mathbb{E}}[e^{r(Y_T-Y_t)-q^2(\langle Y\rangle_T-\langle Y\rangle_t)/2}|\mathcal{F}_t] = \tilde{\mathbb{E}}[e^{(r^2-q^2)(\langle Y\rangle_T-\langle Y\rangle_t)/2}|\mathcal{F}_t]$$

for $r \in (-1/2, 1/2)$. The cases $r = \pm 1/2$ are also included by taking limits, using dominated convergence for the left hand side and monotone convergence for the right.

Translating the above into an equality for the original measure $\mathbb{P}$ yields

$$\mathbb{E}[e^{(1/2+r)(Y_T-Y_t)-q^2(\langle Y\rangle_T-\langle Y\rangle_t)/2}|\mathcal{F}_t] = \mathbb{E}[e^{(Y_T-Y_t)/2+(r^2-q^2)(\langle Y\rangle_T-\langle Y\rangle_t)/2}|\mathcal{F}_t].$$

Evaluating the above equation first at $r = -1/2$ and then at $r = \alpha - 1/2$ and $q^2 = \beta^2 - \alpha^2 + \alpha$, and rearranging yields

$$\mathbb{E}[e^{\alpha(Y_T-Y_t)-\beta^2(\langle Y\rangle_T-\langle Y\rangle_t)/2}|\mathcal{F}_t] = \mathbb{E}[e^{(\alpha^2-\alpha-\beta^2)(\langle Y\rangle_T-\langle Y\rangle_t)/2}|\mathcal{F}_t]$$

for all $\alpha \in [0, 1]$ and $\beta^2 \ge \alpha^2 - \alpha$, and for all $0 \le t \le T$. This implies that conditional on the sigma-field $\mathcal{F}_t^T = \mathcal{F}_t \vee \sigma(\langle Y\rangle_T)$, the increment $Y_T - Y_t$ is normal with mean $-(\langle Y\rangle_T - \langle Y\rangle_t)/2$ and variance $\langle Y\rangle_T - \langle Y\rangle_t$. If we let $X_t = Y_t + \langle Y\rangle_t/2$, then $\langle X\rangle_T = \langle Y\rangle_T$, so the conditional distribution of X_T given $\mathcal{F}_t^T$ is $N(X_t, \langle X\rangle_T - \langle X\rangle_t)$. Hence X is arithmetically symmetric by Theorem 2.2. □

Remark 3. The measures $\hat{\mathbb{P}}$ and $\tilde{\mathbb{P}}$ appearing in above proofs have been studied before in the financial mathematics literature. For instance, the measure $\hat{\mathbb{P}}$ was introduced in Geman, El Karoui, and Rochet [7] and was the object of interest in Schroder [15]; more recently, the measure $\tilde{\mathbb{P}}$ appeared in Theorem 2.2 of Carr and Lee's paper [3].

4. Symmetric and initially flat implies Black–Scholes

In this section, we consider the implications of Theorem 3.1 on market models of option prices. Let the continuous martingale S model the price of a stock after passing from the objective measure to the risk-neutral measure $\mathbb{P}$. For simplicity, we assume that the interest and dividend rates are zero.

We model the price at time t of a call option with strike K and maturity T by the formula

$$C(t, T, K) = \mathbb{E}[(S_T - K)^+|\mathcal{F}_t].$$

In particular, since the stock price and all the call prices are martingales, there is no arbitrage in the market.

It is well-known that call price surface specified in this way satisfies two key properties:

- $T \mapsto C(t, T, K)$ is increasing, with $C(t, t, K) = (S_t - K)^+$ and $C(t, \infty, K) \le S_t$
- $K \mapsto C(t, T, K)$ is decreasing and convex, with $C(t, T, 0) = S_t$ and $C(t, T, \infty) = 0$

We may define random field F of normalized call prices indexed by time to maturity and moneyness defined by

$$F_t(\tau, m) = \mathbb{E}\left[\left(\frac{S_{t+\tau}}{S_t} - m\right)^+ \big| \mathcal{F}_t\right] \tag{5}$$

so that $C_t(T, K) = S_t\ F_t(T - t, K/S_t)$. If we consider a set of arbitrage-free normalized call price surfaces $f : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, 1]$ defined by

$$\begin{aligned}\mathcal{C} \ = \ & \{f : f(\cdot, m) \text{ increasing}, f(0, m) = (1 - m)^+, f(\infty, m) \leq 1,\\ & f(\tau, \cdot) \text{ convex}, f(\tau, 0) = 1, f(\tau, \infty) = 0\}\end{aligned}$$

then the process $(F_t)_{t\geq 0}$ takes values in $\mathcal{C}$. Note that the set $\mathcal{C}$ is convex.

We now consider the effect of the assumption that S is geometrically symmetric. In this case note the equality

$$\begin{aligned}F_t(\tau, m) &= \mathbb{E}\left[\frac{S_{t+\tau}}{S_t}\left(\frac{S_t}{S_{t+\tau}} - m\right)^+ \big| \mathcal{F}_t\right]\\ &= \mathbb{E}\left[\left(1 - m\frac{S_{t+\tau}}{S_t}\right)^+ \big| \mathcal{F}_t\right]\\ &= \mathbb{E}\left[1 - m\frac{S_{t+\tau}}{S_t} + m\left(\frac{S_{t+\tau}}{S_t} - 1/m\right)^+ \big| \mathcal{F}_t\right]\\ &= 1 - m + mF_t(\tau, 1/m).\end{aligned}$$

Hence, we can consider the following subset of normalized call prices that also satisfy put-call symmetry:

$$\mathcal{C}^{\text{sym}} = \{f \in \mathcal{C} : mf(\tau, 1/m) + 1 - m = f(\tau, m)\}.$$

Indeed, if $(F_t)_{t\geq 0}$ takes values in $\mathcal{C}^{\text{sym}}$ then S is geometrically symmetric. As the put-call parity constraint is linear, the set $\mathcal{C}^{\text{sym}}$ is also convex.

The following theorem says that if $(F_t)_{t\geq 0}$ is such that the initial surface F_0 agrees with that of a constant volatility Black–Scholes model, and if the dynamics are constrained to stay in $\mathcal{C}^{\text{sym}}$ for all $t \geq 0$, then the dynamics are trivial $dF_t = 0$.

Theorem 4.1. *Suppose F_t is defined by equation (5) and that $F_0(\tau, m) = \mathrm{BS}(\sigma_0^2\tau, m)$ for all $\tau > 0, m > 0$. If $(F_t)_{t\geq 0}$ is valued in $\mathcal{C}^{\text{sym}}$, then $F_t = F_0$ almost surely for all $t \geq 0$.*

Proof. Since $F_0 = \mathrm{BS}(\sigma_0^2\cdot, \cdot)$ the unconditional distribution of $\log(S_t/S_0)$ is normal with mean $-\sigma_0^2 t/2$ and variance $\sigma_0^2 t$ implying

$$\mathbb{E}\left[\left(\frac{S_t}{S_0}\right)^p\right] = e^{p(p-1)\sigma_0^2 t/2}$$

for all real p. On the other hand, since $(F_t)_{t\geq 0}$ takes values in $\mathcal{C}^{\text{sym}}$, the martingale S is geometrically symmetric and hence of the form $S_t = S_0 e^{X_t - \langle X\rangle_t/2}$ for arithmetically symmetric X. By conditioning on $\langle X\rangle_t$, we have

$$\mathbb{E}\left[\left(\frac{S_t}{S_0}\right)^p\right] = \mathbb{E}[e^{p(p-1)\langle X\rangle_t/2}].$$

In particular, $\mathbb{E}[e^{\lambda\langle X\rangle_t}] = e^{\lambda\sigma_0^2 t}$ for all $\lambda > -1/8$, implying $\langle X\rangle_t = \sigma_0^2 t$ almost surely, proving the claim. $\square$

Remark 4. Recall that we define the implied volatility $\Sigma_t(\tau, m)$ implicitly by the formula

$$F_t(\tau, m) = \mathrm{BS}[\tau\Sigma_t(\tau, m), m].$$

where the function BS is defined in equation (3). Since BS is in $\mathcal{C}^{\mathrm{sym}}$, we see that the geometric symmetry of S implies the symmetry of the implied volatility surface since $F_t(\tau, 1/m) = \mathrm{BS}[\tau\Sigma_t(\tau, m)^2, 1/m]$ and hence $\Sigma_t(\tau, m) = \Sigma_t(\tau, 1/m)$. Theorem 4.1 says that if the initial implied volatility surface is flat, in the sense that $\Sigma_0(\tau, m) = \sigma_0$ for all $\tau > 0, m > 0$, and if the surface is assumed to be symmetric for all future times, then the stock price S must be given by the Black–Scholes model.

Remark 5. Another way to view Theorem 4.1 is as follows: if S is geometrically symmetric, then the random field F_t has the integral representation

$$F_t(\tau, m) = \int_{[0,\infty)} \mathrm{BS}(v, m)\mu_t(\tau, dv)$$

where the random measure $\mu_t(\tau, \cdot)$ is a regular conditional distribution of the increment $\langle \log S\rangle_{t+\tau} - \langle \log S\rangle_t$. Theorem 4.1 says that if the initial measures $\mu_0(\tau, \cdot) = \delta_{\sigma_0^2\tau}$ are Dirac point-masses, then the stochastic dynamics of this family of measures are trivial.

Remark 6. There has been significant recent interest in applying the ideas of Heath, Jarrow, and Morton [9] to equity markets. An early paper is this direction is by Schönbucher [14]; see the papers of Carmona and Nadtochiy [2], Jacod and Protter [10], and Schweizer and Wissel [16, 17] for more recent advances.

Indeed, for the purposes of hedging exotic options with portfolios of calls, it is necessary to have the joint dynamics of all the call prices available. In particular, suppose that the stock price is given by

$$dS_t = S_t\sigma_t \cdot dW_t.$$

for a possibly multi-dimensional Brownian motion W and predictable volatility process σ. Under some smoothness assumptions on the field $(F_t)_{t\geq 0}$, the generalized Itô formula and the fact that $C_t(T, K) = S_t F_t(T - t, K/S_t)$ defines a martingale together imply

$$dF_t(\tau, m) = \left(\frac{\partial F_t}{\partial \tau} - \frac{1}{2}m^2\frac{\partial^2 F_t}{\partial m^2}|\sigma_t|^2 + m\frac{\partial B_t}{\partial m}\cdot\sigma_t - B_t\cdot\sigma_t\right)dt + B_t(\tau, m)\cdot dW_t \tag{6}$$

where the random field B is related to martingale representation of $(C_t(T, K))_{t\geq 0}$. For instance, if S is in the domain of the Malliavin derivative operator D, then the volatility B is given by the formula

$$B_t(\tau, m) = \left(m\frac{\partial F_t}{\partial m} - F_t\right)\sigma_t + \mathbb{E}\left[\frac{D_t S_{t+\tau}}{S_t}\mathbb{1}_{\{S_{t+\tau}/S_t > m\}}\middle|\mathcal{F}_t\right]$$

by the Clark–Ocone formula.

The so-called market model approach to this issue is essentially to invert the above discussion. That is, one takes the time-0 normalized call price surface F_0 as the given initial condition, and evolves the surface $(F_t)_{t\geq 0}$ in such a way as to prohibit arbitrage. In particular, it would be convenient to have easy-to-check sufficient conditions on the process σ and random field B such that equation (6) has an $\mathcal{C}$-valued solution. Indeed, if $(F_t)_{t\geq 0}$

were such a solution and if $S_t F(T-t, K/S_t)$ defines a martingale (not just a local martingale) for each $T>0$ and $K>0$, then we could take $C(t,T,K) = S_t F(T-t, K/S_t)$ since $C(T,T,K) = (S_T - K)^+$ by the definition of $\mathcal{C}$, and hence

$$C(t,T,K) = \mathbb{E}[(S_T - K)^+ | \mathcal{F}_t].$$

Theorem 4.1 can be taken to be bad news for the success of the above program: if we insist that the solution to (6) takes value in the set $\mathcal{C}^{\text{sym}} \subset \mathcal{C}$, then the admissible random fields B are strongly dependent on the initial condition F_0. In particular, if $F_0 = \text{BS}(\sigma_0^2 \cdot, \cdot)$ then $B_t(\tau, m) = 0$ identically.

Following Remark 2, it would be interesting to know if there exists continuous martingales, other than geometric Brownian motion, whose marginal distributions are log-normal. If not, the above theorem could be strengthened by allowing the process $(F_t)_{t\geq 0}$ to evolve in $\mathcal{C}$ rather than the much smaller set $\mathcal{C}^{\text{sym}}$. In other words, if the initial implied volatility surface is flat, and the asset price dynamics are continuous, must the implied volatility surface remain flat for all future times?

5. Conclusion

We have seen how the assumption that a continuous local martingale has conditionally symmetric increments implies a strong structural property. In particular, Brownian motion scaled by constants are seen to be the only extremal distributions.

It seems plausible that a similar characterization can be carried through for arithmetically symmetric local martingales with jumps. In fact, Ocone [12] has shown that if a general càdlàg local martingale X has the property that $\int(\mathbb{1}_{[0,s]} - \mathbb{1}_{(s,\infty)})dX$ and X share the same law, then X has conditionally independent increments. However, as mentioned in Section 1, arithmetic symmetry is weaker a priori than Ocone's condition. In any case, it seems unlikely that the method of proof presented here could directly handle the general case.

For the application to finance, the general characterization of geometric symmetry is also of interest. Fajardo and Mordecki [6] showed that an exponential Lévy process is geometrically symmetric its Lévy measure ν satisfies $\nu(-dy) = e^y \nu(dy)$. Carr and Lee [3] built more examples of geometrically symmetric process by introducing random time changes to a family of exponential Lévy processes. It is an open question if these are the only examples.

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