AN EQUILIBRIUM MODEL OF MARKET EFFICIENCY WITH BAYESIAN LEARNING: EXPLICIT MODES OF CONVERGENCE TO RATIONAL EXPECTATIONS EQUILIBRIUM IN THE PRESENCE OF NOISE TRADERS

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ABSTRACT. A simple discrete-time financial market model is introduced. The market participants consist of a collection of noise traders as well as a distinguished agent who uses the price information as it arrives to update her demand for the assets. It is shown that the distinguished agent's demand converges, both almost surely and in mean square, to a demand consistent with the rational expectations hypothesis, and the rate of convergence is calculated explicitly. Furthermore, the convergence of the standardised deviations from this limit is established. The rate of convergence, and hence the efficiency of this market, is an increasing function of both the risk-free interest rate and the relative number of noise traders in the market. An efficient market, therefore, measured in terms of a high proportion of informed traders, seems incompatible with the notion that efficient markets converge quickly.

1. Introduction

This short note introduces a simple discrete-time model of a market. As in many economic models, the market consists of assets and agents, and the transaction price of the assets each period is determined by equating the aggregate demand of the agents with the total supply. An important feature of the model is that there is one distinguished agent who uses the market information as it arrives to update her demand for the assets. The remaining agents are noise traders. We will see that in this simple model, the distinguished agent's demand converges to a demand consistent with the rational expectations hypothesis. This convergence is in both the almost sure and the mean square sense. Furthermore, in this model, the rate of convergence can be calculated explicitly. The rate depends on two parameters: the risk-free interest rate and the relative size of the distinguished agent in the market. We further investigate the convergence of the standardised deviations from this limit.

Although most models do not require that all investors be rational or well informed, a reasonable intuition about an efficient market is that a majority of agents are indeed rational and also that prices adjust rapidly. One of our contributions to the literature is to show that the higher the proportion of noise traders, the faster the rate of convergence to rational expectations equilibria. This conclusion seems at odds with the previous held intuition.

The notion that market efficiency is tied to the speed at which markets incorporate new information is fundamental. Indeed, Fama [10, p. 39] states that "on the average the full effects of new information on intrinsic values will be reflected nearly instantaneously in actual

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prices". Chordia et al [6] provide empirical evidence on the speed of convergence on the New York stock exchange. They do this by examining patterns of serial dependence over different time intervals. This approach has been followed by subsequent authors.

A closely related concept, that of price discovery, is defined by Lehmann [12, p. 259] as "the efficient and timely incorporation of the information implicit in investor trading into market prices." Similar notions include such ideas as information efficiency which refers to the ability of a market to incorporate new information into equilibrium prices in a timely manner.

There is an existing literature on the rate of convergence based on learning models in rational expectations which have some similarities to our work. For instance, the papers of Bray [1] and of Margaritis [13] establish almost convergence in models with least squares learning. In some cases, researchers have resorted to simulation to assess rates of convergences; see Bray & Savin [2] for example. We note the paper of Chevillon & Mavroeidis [5] where explicit rates of convergence are calculated. See also the paper of Vives [14] who provides references to earlier work.

There are many papers that capture features of our market structure. There are a number of models that provide evidence on the necessity in an efficient market for the presence of both informed traders and noise traders, an interesting early reference being Cornell & Roll [8]. We assume our distinguished agent is a Bayesian. A similar approach was adopted by Chakrabarti & Roll [3]. Many features of our structure bear resemblance to the work of De Long et al [9].

The remainder of the paper is organised as follows. In section 2 the modelling framework and notation is established. In section 3 the main mathematical results are presented. In particular, we show that when an agent adaptedly calculates her demand for a risky asset by a Bayesian updating rule, the posterior mean converges to the rational expectation. Furthermore, the rate of convergence is calculated explicitly, highlighting the role of the model parameters. Finally, the convergence of the standardised deviations from the limit is established. In section 4 the proofs of the main results are presented along with some interesting auxiliary results. Section 5 concludes.

2. The model set-up

In this section, the modelling framework is established and the notation is introduced.

2.1. **The assets.** We work in a discrete-time financial market consisting of two traded assets, a bond and a stock.

The first asset is a risk-free bond. It pays an interest payment of r units of money at the end of each period, where the positive constant r is known to all market participants. This asset is effectively of infinite supply, so that there is no constraint on the number of shares of this asset available to trade in the market. In particular, we assume that the price in each period is a constant, which we take to be one unit of money. These assumptions are intended to model a very large, safe borrower such as the US Treasury.

The second asset is a risky stock. In principle, the riskiness of this asset comes from two sources: both the future dividend payouts and the future stock prices appear random to the market participants. Indeed, unlike the safe asset, we assume that the stock is of finite net supply of exactly K shares. In particular, the price is set each period in such a way that total demand equals total supply. We will let P_t denote its price at time t. To simplify the analysis, we focus our attention on the randomness arising from these price movements and we assume that the dividend payment is actually a constant d which is known to the market participants.

2.2. **The agents.** In this market, we introduce a collection of agents. We will soon specialise to the case where there are two representative agents. The first is a distinguished agent, labelled L, who we assume models the market, and tries to learn the parameters of her model by a simple Bayesian updating of her prior beliefs as market information arrives. The second is the rest of the market, labelled R, whose demand function we will model exogeneously.

We first model the agents individually. Each agent must decide how many shares of the risky asset to hold during each period. In order to specify each agent's demand function, we assume that at time t agent i models the price P_{t+1} as a random variable with mean $\mu_{t,i}$ and variance σ_i^2. In particular, we allow the agent's subjective conditional expectation $\mu_{t,i}$ of the future price P_{t+1} to depend on the current calendar date t, but we insist for the sake of simplicity that the subjective conditional variance σ_i^2 be independent of time t.

To model the agents' demand for the stock, we assume that she tries to maximise the mean–variance objective

$$\mathbb{E}_{t,i}(X_{t+1}) - \frac{\gamma_i}{2}\mathrm{Var}_{t,i}(X_{t+1})$$

where γ_i is a positive constant measuring agent i's risk aversion, X_{t+1} is the value of her portfolio at time $t+1$ and $\mathbb{E}_{t,i}$ and $\mathrm{Var}_{t,i}$ denote the time t her subjective conditional expectation and variance, respectively. Note that with zero cost an agent can borrow P_t shares of the bond at time t and simultaneously buy one share of the stock. This portfolio is worth

$$d + P_{t+1} - (1+r)P_t$$

which is simply the dividend payment plus the resale value of the risky asset minus the return on the bond. Hence she picks her holding H_t to maximise

$$[d + \mu_{t,i} - (1+r)P_t]H_t - \frac{\gamma_i}{2}\sigma_i^2 H_t^2.$$

Maximising the quadratic yields

$$H_t^* = \frac{1}{\gamma_i \sigma_i^2}[d + \mu_{t,i} - (1+r)P_t].$$

We rewrite the above equation in terms of the agent's demand function $D_{t,i}(\cdot)$. In particular, if the stock price at time t equals p, the agent would want to hold $D_{t,i}(p)$ shares where

$$D_{t,i}(p) = \frac{1}{\gamma_i \sigma_i^2}[d + \mu_{t,i} - (1+r)p].$$

As mentioned above, we distinguish one agent, labelled L, and aggregate the demand of the rest of the market, labelled R, to arrive at

$$D_{t,i}(p) = \alpha_i[d + \mu_{t,i} - (1+r)p] \tag{1}$$

where i is either L or R, and the effective parameters are calculated as

$$\alpha_{\mathrm{L}} = \frac{1}{\gamma_{\mathrm{L}}\sigma_{\mathrm{L}}^2} \tag{2}$$

$$\alpha_{\mathrm{R}} = \sum_{i\neq \mathrm{L}} \frac{1}{\gamma_i\sigma_i^2} \tag{3}$$

and

$$\mu_{t,\mathrm{R}} = \frac{1}{\alpha_{\mathrm{R}}} \sum_{i\neq \mathrm{L}} \frac{1}{\gamma_i\sigma_i^2}\mu_{t,i} \tag{4}$$

We consider the parameters α_{L} as a quantification of agent L's risk tolerance. Note that it increases when either her risk aversion γ_{L} or her subjective variance σ_{L}^2 of the risky asset return decreases. Since this model allows us to aggregate sub-agents into agents by aggregating their risk tolerances, we can interpret, assuming that each sub-agent has control over the same amount of investible wealth, that the parameters α_{L} and α_{R} are proxies for the market share of agents L and R respectively.

To further specify the model, we need to know how the agents form their expectations $\mu_{t,i}$ of future prices as time progresses. As mentioned above, we will assume that the subjective expectation $\mu_{t,\mathrm{R}}$ of the rest of the market is exogeneously given. We suppose that there exists an objective probability measure $\mathbb{P}$, and that each $\mu_{t,\mathrm{R}}$ is a random variable. The idea is that we will focus on the distinguished agent L, and do not explicitly model how the rest of the market formulates its expectations of the evolution of the stock price. By introducing randomness, we also allow for the trading that is a result of tax policies, natural disasters, inside information, intuition and other complications.

On the other hand, how the distinguished agent calculates her subjective expectation $\mu_{t,\mathrm{L}}$ is the subject of section 2.3 below.

2.3. **Bayesian learning.** We now consider the problem from the point of view of the distinguished agent labelled L. In order to specify her demand function $D_{t,\mathrm{L}}(\cdot)$ at time t, we need to model her subjective mean $\mu_{t,\mathrm{L}}$ of the future stock price. We assume that she has a statistical model for the sequence of prices $(P_t)_{t\geq 1}$. Since at time t she can observe the prices $(P_s)_{1\leq s\leq t}$ she will use these data to estimate the parameters of her model.

For the sake of tractability, we suppose that under the agent's subjective probability distribution, the sequence $(P_t)_{t\geq 1}$ is independent and identically distributed $N(\mu, \sigma_{\mathrm{L}}^2)$ random variables. The agent knows the variance parameter σ_{L}^2 with certainty, but is possibly uncertain about the mean μ. Her prior (time-0) distribution for μ is $N(\mu_{0,\mathrm{L}}, v_0)$ where $\mu_{0,\mathrm{L}}$ and v_0 are fixed parameters.

After observing $P_1, \dots, P_t$ her posterior distribution of μ is $N(\mu_{t,\mathrm{L}}, v_t)$ where

$$\mu_{t,\mathrm{L}} = \frac{\sum_{u=1}^t P_u + a\mu_{0,\mathrm{L}}}{t+a} \tag{5}$$

and

$$v_t = \frac{\sigma_{\mathrm{L}}^2}{a+t} \tag{6}$$

where

$$a = \frac{\sigma_{\mathrm{L}}^2}{v_0} \tag{7}$$

is the ratio of the variance of stock price to variance of the prior distribution for the mean of the stock price for agent L. It is important to notice that agent L uses a deterministic rule to convert the observed market prices into an estimate of the mean of future price.

2.4. **Price formation via market clearing.** We assume that prices are set by market clearing, and hence total demand must equal total supply:

$$D_{t,\mathrm{L}}(P_t) + D_{t,\mathrm{R}}(P_t) = K,$$

where K is the total number of shares of the risky asset. Substituting the assumed form of the demand functions from equation (1) yields

$$P_t = \frac{1}{1+r}[\theta\mu_{t,\mathrm{L}} + (1-\theta)\mu_{t,\mathrm{R}} + d - k] \tag{8}$$

where

$$k = \frac{K}{\alpha_{\mathrm{L}} + \alpha_{\mathrm{R}}} \tag{9}$$

is the total supply normalised by a measure of the total market risk tolerance and

$$\theta = \frac{\alpha_{\mathrm{L}}}{\alpha_{\mathrm{L}} + \alpha_{\mathrm{R}}} \tag{10}$$

measures the risk tolerance of the distinguished agent L relative to the aggregate risk tolerance of the entire market. As we discussed earlier, the parameters α_i can be interpreted as the risk-tolerance weighted market share of each agent i, and hence θ is the relative market share of agent L.

2.5. **Summary of parameters.** Before continuing to the analysis of this model, we pause briefly to list the model parameters and their significance. Our analysis will depend on the following effective parameters

- $\mu_{0,\mathrm{L}}$, agent L's initial estimate of the mean of next period's stock price,
- a, the ratio of agent L's variance of next period's stock price to her initial uncertainty of its mean,
- θ, the risk-tolerance weighted relative market share of agent L,
- r, the risk-free interest rate,
- d, the constant stock dividend,
- k, a normalised total supply of the stock,
- the distribution of the random variables $\mu_{t,\mathrm{R}}$, agent R's subjective mean of next period's stock price.

These parameters can be built out of more fundamental parameters. For instance,

- σ_{L}^2, agent L's subjective variance of next period's stock price,
- v_0, the variance of agent L's prior distribution for the subjective mean,

and the effective parameter a is defined by $a = \sigma_{\mathrm{L}}^2/v_0$.

Similarly, with the more fundamental parameters

- γ_i, agent i's risk aversion parameter,

- σ_i^2, agent i's subjective variance of next period's stock price,
- K, the total number of shares of stock outstanding

we can define the parameters $\alpha_{\mathrm{L}} = 1/(\gamma_{\mathrm{L}}\sigma_{\mathrm{L}}^2)$, and $\alpha_{\mathrm{R}} = \sum_{i\neq \mathrm{L}} 1/(\gamma_i\sigma_i^2)$, as well as $k = K/(\alpha_{\mathrm{L}}+\alpha_{\mathrm{R}})$ and $\theta = \alpha_{\mathrm{L}}/(\alpha_{\mathrm{L}}+\alpha_{\mathrm{R}})$.

3. The main results

In this section we present our main results concerning the long time behaviour of this model. All proofs will be given in Section 4 below.

Recall that randomness enters the model exogeneously via the sequence $(\mu_{t,\mathrm{R}})_{t\geq 1}$ of subjective means of the noise traders. Our results will depend on the distributional properties of this sequence under the objective probability measure. Henceforth, we will operate under the following additional assumption:

Assumption 3.1. *The random variables $(\mu_{t,\mathrm{R}})_{t\geq 1}$ are independent, identically distributed with mean $\mathbb{E}(\mu_{\mathrm{R}})$ and finite variance $\mathrm{Var}(\mu_{\mathrm{R}})$ with respect to the objective probability measure $\mathbb{P}$.*

3.1. Convergence to rational expectations.

A natural notion of rationality in this context is when an agent's subjective conditional expectation of next period's stock price agrees with the objective conditional expectation. In particular, we will say that agent i is rational if

$$\mu_{t,i} = \mathbb{E}_t(P_{t+1}) \text{ for all } t \geq 1$$

where $\mathbb{E}_t$ denotes the conditional expectation computed with respect to the objective probability measure $\mathbb{P}$ given the price history $P_1, \ldots, P_t$. This notion of rationality is very strong, and there is no a priori reason to hope that is should hold for any agent in this model.

However, there is another, weaker notion of rationality which is appropriate here. We will say that agent i is asymptotically rational if

$$|\mu_{t,i} - \mathbb{E}_t(P_{t+1})| \to 0 \text{ as } t \to \infty$$

in some sense. We will see that the distinguished agent L is indeed asymptotically rational in a strong sense. The first result in this direction is the following proposition:

Proposition 3.2. *We have*

$$\mu_{t,\mathrm{L}} - \mathbb{E}_t(P_{t+1}) = \varepsilon\left(1 + \frac{1-\varepsilon}{t+a+\varepsilon}\right)(\mu_{t,\mathrm{L}} - \bar{\mu})$$

for all $t \geq 0$, where

$$\bar{\mu} = \frac{(1-\theta)\mathbb{E}(\mu_{\mathrm{R}}) + d - k}{1+r-\theta}. \tag{11}$$

and

$$\varepsilon = 1 - \frac{\theta}{1+r}. \tag{12}$$

Note that since the risk-tolerance weighted market share θ of agent L is less than one and the interest rate r is positive, the parameter ε is positive, and in particular, not zero. As a consequence, the rationality or asymptotic rationality of agent L is determined by the difference $\mu_{t,\mathrm{L}} - \bar{\mu}$. Therefore, the message of Proposition 3.2 is that our investigation of the asymptotic rationality should focus on the long time behaviour of $\mu_{t,\mathrm{L}}$. We will see that the parameters $\bar{\mu}$ and ε are key to our analysis. We will explore their economic significance below.

Remark 1. Before proceeding, we recall various notions of convergence of random variables. We are given a sequence $(X_t)_{t\geq 0}$ of random variables and a random variable X. Perhaps the most intuitive notion of convergence is that of almost sure convergence. We say $X_t \to X$ almost surely if $\mathbb{P}(X_t \to X) = 1$. That is, for almost every outcome ω in the sample space Ω, the sequence of real numbers $X_t(\omega)$ converges to the real number $X(\omega)$.

Another notion of convergence is convergence in L^2 or mean-square. We say $X_t \to X$ in L^2 if $\mathbb{E}(X^2) < \infty$ and $\mathbb{E}(|X_t - X|^2) \to 0$. Unlike the notion of almost sure convergence, convergence in L^2 involves averaging over the sample space. In fact, these two notions of convergence are not directly comparable in the sense that one does not necessarily imply the other.

For instance, let $A_1, A_2, \ldots$ be a decreasing sequence of events, in the sense that $A_{t+1} \subseteq A_t$. Suppose that $\mathbb{P}(A_t) = 1/t$ and that $X_t = \sqrt{t}\mathbf{1}_{A_t}$, where $\mathbb{1}_A$ denotes the indicator of the event A. Then $X_t \to 0$ almost surely, but since $\mathbb{E}(X_t^2) = 1$ for all t, the sequence $(X_t)_{t\geq 1}$ does not converge in L^2.

Now let $A_1, A_2, \ldots$ be *independent* events with $\mathbb{P}(A_t) = 1/t$, and let $X_t = \mathbb{1}_{A_t}$. Since $\mathbb{E}(X_t^2) = 1/t \to 0$, we see that $X_t \to 0$ in L^2. However, since $\sum_{t\geq 1} \mathbb{P}(A_t) = +\infty$, we have $\mathbb{P}(X_t = 1 \text{ infinitely often}) = 1$ by the second Borel–Cantelli lemma, and hence $(X_t)_{t\geq 1}$ does not converge almost surely.

Finally, we say $X_t \to X$ in distribution if $F_t(x) \to F(x)$ for all points x of continuity of F, where F_t and F are the cumulative distribution functions of X_t and X, respectively. This notion of convergence is weaker than the other two, in the sense that if $X_t \to X$ either almost surely or in L^2, then $X_t \to X$ in distribution.

On the other hand, if $X_t \to X$ in distribution, it does not necessarily follow that $X_t \to X$ either almost surely or in L^p. For instance, let A be an event with $\mathbb{P}(A) = 1/2$, and let $X_t = \mathbb{1}_A$ for all $t \geq 1$ and $X = 1 - \mathbb{1}_A = \mathbf{1}_{A^c}$. Then X_t and X have the same distribution and hence $X_t \to X$ in distribution. But since $|X_t - X| = 1$, the sequence $(X_t)_{t\geq 1}$ does not converge in either of the stronger senses described above.

We can now state our main result on asymptotic rationality. It is in the spirit of similar results of Bray [1] and Margaritis [13] for models of least squares learning.

Theorem 3.3. *We have*

$$\mu_{t,\mathrm{L}} \to \bar{\mu} \textit{ almost surely and in } L^2$$

at $t \to \infty$*.*

The convergence in L^2 is interesting mathematically, but the real economic interest of Theorem 3.3 is the almost sure convergence. Indeed, an economic agent only experiences one history of the world ω, which in this model can be identified with the sequence $(\mu_{t,\mathrm{R}}(\omega))_{t\geq 1}$ of realised subjective means of the noise trader. The almost sure convergence implies that with probability one, for this particular outcome ω, the sequence $(\mu_{t,\mathrm{L}}(\omega))_{t\geq 1}$ of agent L's realised subjective means converges to a constant $\bar{\mu}$ which does not depend on ω.

We now comment on the economic significance of the limit $\bar{\mu}$. Note that

$$\bar{\mu} = \frac{d-k}{r} + \frac{1-\theta}{1+r-\theta}\left(\mathbb{E}(\mu_{\mathrm{R}}) - \frac{d-k}{r}\right).$$

We can interpret the quantity $(d-k)/r$ as the common subjective mean such that all market participants are rational. Indeed, note that $\mathbb{E}(\mu_{\mathrm{R}}) = (d-k)/r$ if and only if $\bar{\mu} = (d-k)/r$. In this sense, we may regard $(d-k)/r$ as the universal rational expectation. In particular, the limit $\bar{\mu}$ is the sum of the universal rational expectation plus a term reflecting the bias of the noise traders away from this universal rational expectation. The coefficient $\frac{1-\theta}{1+r-\theta}$ of this bias term is a decreasing function of the parameter θ, the risk-tolerance weighted relative market share. In particular, as θ increases to one, the effect of the noise traders on the limit $\bar{\mu}$ vanishes.

Recall that the parameter k can be expressed as

$$k = \frac{K}{\sum_i \frac{1}{\gamma_i \sigma_i^2}}.$$

Note that k is increasing in the risk aversion parameters γ_i and the subjective variances σ_i^2, and in particular, that $k = 0$ if $\gamma_i = 0$ or $\sigma_i^2 = 0$ for some agent i. Therefore we may regard $d-k$ as a risk-adjusted dividend, and since we have the identity

$$\frac{d-k}{r} = \sum_{t=1}^{\infty} (d-k)(1+r)^{-t}$$

the universal rational expectation $(d-k)/r$ is the present discounted value of the stream of risk-adjusted dividend payments.

3.2. **The Rate of convergence.** An interesting feature of our analysis is that the rate of convergence to rationality can be calculated explicitly. We will see that the rate of convergence is determined by the parameter ε defined by equation (12). Note that ε relates the relative market share θ of agent L to the interest rate r on the risk-free bond. Note that since $r > 0$ and $0 < \theta < 1$, we have the bound $0 < \varepsilon < 1$. Furthermore, note that ε increases with r but decreases with θ. We will see that the long time behaviour is very different, depending on whether $0 < \varepsilon < 1/2$, $\varepsilon = 1/2$ or $1/2 < \varepsilon < 1$.

To state our next result, let

$$\varsigma_t^{(\varepsilon)} = \begin{cases} t^{-\varepsilon} & \text{if } 0 < \varepsilon < 1/2 \\ t^{-1/2}\sqrt{\log t} & \text{if } \varepsilon = 1/2 \\ \frac{t^{-1/2}}{\sqrt{2\varepsilon - 1}} & \text{if } \varepsilon > 1/2. \end{cases} \tag{13}$$

Theorem 3.4. *We have*

$$\mathbb{E}(\mu_{t,\mathrm{L}}) = \bar{\mu} + O(t^{-\varepsilon})$$

and

$$\mathrm{Var}(\mu_{t,\mathrm{L}}) = O((\varsigma_t^{(\varepsilon)})^2)$$

as $t \to \infty$.

We note that the computation of the variance in Theorem 3.4 above is a special case of a result of Chevillon & Mavroeidis [5, Theorem 2].

An intuitive explanation for why the Bayesian trader's subjective expectation may converge to the rational expectation $\bar{\mu}$ more slowly than the usual $t^{-1/2}$ rate is that the agent's own trading dilutes the signal encoded in the prices. For instance, when θ is close to one, meaning that the risk tolerance of agent L is large relative to the market, then a great deal of the price movement can be explained by agent L's trading activity. Therefore, it is not surprising that it takes a long time for her to learn the parameters of the noise traders' random demand. Similarly, when the risk-free interest rate r is small, the trader is induced to have a comparatively larger holding of the risky asset, and this trading activity has a larger impact on the prices.

The rate of convergence is maximal whenever $\varepsilon > 1/2$. Expressing this inequality in terms of the relative market share θ and interest rate r yields

$$2\theta < 1 + r.$$

When expressed in terms of the more fundamental risk aversion parameters γ_i and subjective variances σ_i^2 we have that the convergence happens at the maximal rate if

$$\frac{1-r}{\gamma_{\mathrm{L}}\sigma_{\mathrm{L}}^2} < \sum_{i \neq \mathrm{L}} \frac{1+r}{\gamma_i \sigma_i^2}.$$

3.3. **Deviations from the limit.** Now that we have established the limit and the rate of convergence, we focus our attention on the sequence of standardised deviations from the limit

$$\Delta_t = \frac{\mu_{t,\mathrm{L}} - \bar{\mu}}{\varsigma_t^{(\varepsilon)}}. \tag{14}$$

The next theorem establishes the convergence of the sequence $(\Delta_t)_{t \geq 1}$ to some random variable Δ. This convergence validates the approximation

$$\mu_{t,\mathrm{L}} \approx \bar{\mu} + \varsigma_t^{(\varepsilon)} \Delta,$$

which in light of Proposition 3.2 implies

$$\mu_{t,\mathrm{L}} - \mathbb{E}_t(P_{t+1}) \approx \varepsilon \varsigma_t^{(\varepsilon)} \Delta$$

for t large. However, the interpretation of the above approximations depends crucially on the mode of convergence. We will see that the cases $0 < \varepsilon < 1/2$ and $1/2 \leq \varepsilon < 1$ are very different in this respect.

Theorem 3.5. *If $0 < \varepsilon < 1/2$ then there is a random variable Δ such that*

$$\Delta_t \to \Delta \textit{ almost surely and in } L^2$$

as $t \to \infty$. The mean of Δ is

$$\mathbb{E}(\Delta) = \frac{\Gamma(a+\varepsilon)}{\Gamma(a)} (\mu_{0,\mathrm{L}} - \bar{\mu})$$

where Γ is the Gamma function, and the variance is

$$\mathrm{Var}(\Delta) = \left(\frac{1-\theta}{1+r} \right)^2 \mathrm{Var}(\mu_{\mathrm{R}}) \left(\frac{\Gamma(a+\varepsilon)}{\Gamma(a)} \right)^2 \sum_{t=1}^{\infty} \left(\frac{(a+\varepsilon-1)_t}{(a)_t} \right)^2$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol.

If $1/2 \le \varepsilon < 1$ *then there is a normal random variable* Δ *such that*

$$\Delta_t \to \Delta \text{ in distribution}$$

as $t \to \infty$. *The mean of* Δ *is*

$$\mathbb{E}(\Delta) = 0$$

and the variance is

$$\mathrm{Var}(\Delta) = \left(\frac{1-\theta}{1+r}\right)^2 \mathrm{Var}(\mu_{\mathrm{R}}).$$

Remark 2. The Gamma function Γ is defined by the integral

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds$$

for positive real x. Recall that the hypergeometric function ${}_3F_2$ is defined by the series

$${}_3F_2(p, q, r; s, t; z) = \sum_{n=0}^{\infty} \frac{(p)_n (q)_n (r)_n}{(s)_n (t)_n n!} z^n.$$

Hence, for $0 < \varepsilon < 1/2$, the variance can be expressed equivalently as

$$\mathrm{Var}(\Delta) = \left(\frac{\Gamma(a+\varepsilon)(1-\theta)}{\Gamma(a+1)(1+r)}\right)^2 \mathrm{Var}(\mu_{\mathrm{R}})\ {}_3F_2(1, a+\varepsilon, a+\varepsilon; a+1, a+1; 1).$$

Note that when $0 < \varepsilon < 1/2$, the convergence of the normalised difference Δ_t is in a very strong sense. However, the limit Δ depends on agent L's prior beliefs through the parameters $\mu_{0,\mathrm{L}}$ and $a = \sigma_{\mathrm{L}}^2 / v_0$.

On the other hand, when $1/2 \le \varepsilon < 1$, the convergence, as reported in Theorem 3.4, of the standardised difference is a much weaker sense. In this case, we have a central limit-type result, with the limit having mean zero and a variance that does not depend on the agent's prior beliefs. Furthermore, the limit distribution is always normal, regardless of the distribution of the noise traders subjective means $\mu_{t,\mathrm{R}}$.

One may ask when $1/2 \le \varepsilon < 1$, is it possible to replace the convergence in distribution to a stronger notion of convergence. The answer turns out to be no:

Proposition 3.6. *Suppose* $1/2 \le \varepsilon < 1$. *If* $\mathrm{Var}(\mu_{\mathrm{R}}) > 0$ *then the sequence* $(\Delta_t)_{t \ge 1}$ *does not converge almost surely nor does the sequence converge in* L^2.

Similarly, one may ask when $0 < \varepsilon < 1/2$ if the limit random variable is normally distributed. The answer depends on the noise traders subjective means $\mu_{t,\mathrm{R}}$:

Proposition 3.7. *Suppose* $0 < \varepsilon < 1/2$. *Then the limit* Δ *is normally distributed if and only if* $\mu_{t,\mathrm{R}}$ *is normally distributed for each* $t \ge 1$.

3.4. **Prices decorrelate asymptotically.** In the previous section, we have studied the price process as it relates to to agent L's subjective expectations. In this section, we present results concerning the convergence of the mean and autocovariance of the price process itself. In fact, a related result for stationary but not necessarily independent sequences has been provide by Chevillon & Mavroeidis [5, Theorem 2].

Theorem 3.8. *As $t \to \infty$ we have*

$$\mathbb{E}(P_t) = \bar{\mu} + O(t^{-\varepsilon})$$

and

$$\mathrm{Var}(P_t) = \frac{(1-\theta)^2}{(1+r)^2}\mathrm{Var}(\mu_{\mathrm{R}}) + O((\varsigma_t^{(\varepsilon)})^2)$$

where $\varsigma_t^{(\varepsilon)}$ is defined by equation (13)*. Furthermore, for fixed $u \geq 1$ as $t \to \infty$ we have*

$$\mathrm{Cov}(P_t, P_{t+u}) = O((\varsigma_t^{(\varepsilon)})^2)$$

and for fixed $t \geq 1$ as $u \to \infty$ we have

$$\mathrm{Cov}(P_t, P_{t+u}) = O(u^{-\varepsilon}).$$

The rate of convergence of the objective mean $\mathbb{E}(P_t)$ to the rational mean $\bar{\mu}$ is not surprising in light of Proposition 3.2 and Theorem 3.4. Possibly of more interest is when we consider prices at two dates t and $t+u$ separated by u periods; then the prices decorrelate asymptotically. In fact, the decorrelation appears in two different asymptotic regimes. The first regime is when the number u of periods is fixed and we let $t \to \infty$, in which case the autocovariance decays at a rate which depends on the parameter ε. Recall that we have assumed that agent L believes that, conditional on the value of the true mean μ, the random variables $(P_t)_{t\geq 1}$ are independent $N(\mu, \sigma_{\mathrm{L}}^2)$. We see that this Bayesian trader operates under incorrect assumptions, as it is generally not the case that the prices are independent in this model. Nevertheless, Theorem 3.8 shows that these assumptions become more plausible asymptotically since the mean and variance are converging to constants and the autocovariance is converging to zero.

The second regime is when the date t is fixed and we let the number of periods $u \to \infty$. In this case the autocovariance decays precisely at rate ε. Note that the price process in this model exhibits long memory since $\varepsilon < 1$. It is interesting that the long memory property arises naturally from the modelling assumptions with the rate of decay of the autocovariance determined endogeneously, despite the assumption that the dynamics are driven by an independent and identically distributed sequence of exogeneously given random variables $(\mu_{t,\mathrm{R}})_{t\geq 1}$.

3.5. **Second moment rationality.** So far we have used a standard notion of rationality, which we could call rationality in mean or first moment rationality. We could go further. Agent i is second moment rational if she is both first moment rational and that

$$\sigma_i^2 = \mathrm{Var}_t(P_{t+1}),$$

where Var_t is the conditional variance with respect to the objective probability measure $\mathbb{P}$ given the history $P_1, \ldots, P_t$. As before, we can also consider the notion of second moment asymptotic rationality as first moment asymptotic rationality plus the requirement that

$$|\sigma_i^2 - \mathrm{Var}_t(P_{t+1})| \to 0$$

in some sense. We will see that $\mathrm{Var}_t(P_{t+1})$ is not random in our model, so there is no issue with modes of convergence to address.

Proposition 3.9. *We have*

$$\mathrm{Var}_t(P_{t+1}) = \frac{(1-\theta)^2}{(1+r)^2}\mathrm{Var}(\mu_{\mathrm{R}})\left(1+\frac{1-\varepsilon}{t+a+\varepsilon}\right)^2.$$

In particular, agent L is second moment asymptotically rational if and only if

$$\sigma_{\mathrm{L}}^2 = \frac{(1-\theta)^2}{(1+r)^2}\mathrm{Var}(\mu_{\mathrm{R}}) \tag{15}$$

Unfortunately, equation (15) is problematic. Indeed, the right-hand side depends on the market size parameter θ, which in turn depends on the demand sensitivity parameter α_{L}, which finally depends on the subjective variance σ_{L}^2. More explicitly, we can rewrite equation (15) as

$$(1+r)^2\sigma_{\mathrm{L}}^2\left(\sum_i \frac{1}{\gamma_i\sigma_i^2}\right)^2 = \left(\sum_{i\neq \mathrm{L}} \frac{1}{\gamma_i\sigma_i^2}\right)^2\mathrm{Var}(\mu_{\mathrm{R}}).$$

In particular, there is no a priori reason to assume that agent L's subjective variance would satisfy equation (15) and hence, in general, second moment asymptotic rationality does not hold for this model.

4. The proofs

Our analysis of this model begins by relating the sequence $(\mu_{t,\mathrm{L}})_{t\geq 1}$ of agent L's subjective means in terms of the sequence $(\mu_{t,\mathrm{R}})_{t\geq 1}$ of subjective means of the rest of the market.

Proposition 4.1. *For $t \geq 0$, we have*

$$\mu_{t+1,\mathrm{L}} = \frac{t+a}{t+a+\varepsilon}\mu_{t,\mathrm{L}} + \frac{(1-\theta)\mu_{t+1,\mathrm{R}}+d-k}{(1+r)(t+a+\varepsilon)} \tag{16}$$

where ε is given by equation (12).

Proof. Note that by equation (5) we have

$$\mu_{t+1,\mathrm{L}} = \frac{\sum_{u=1}^{t+1} P_u + a\mu_{0,\mathrm{L}}}{t+a+1}.$$

Combining the above expression with equation (5) yields

$$(t+a+1)\mu_{t+1,\mathrm{L}} - (t+a)\mu_{t,\mathrm{L}} = P_{t+1}.$$

The result follows from inserting this formula into equation (8). □

Note that equation (12) expresses agent L's subjective mean recursively. The equation is of the form of a time-inhomogeneous auto-regression where the driving noise term is a function of the subjective means of the noise traders.

Proof of Proposition 3.2. From equations (8) and (19) we see that the sigma-field generated by the random variables $P_1, \ldots, P_t$ and that generated by $\mu_{1,\mathrm{R}}, \ldots, \mu_{t,\mathrm{R}}$ agree. Hence, from equation (8) we have

$$\begin{aligned}\mathbb{E}_t(P_{t+1}) &= \frac{1}{1+r}\left(\theta\mathbb{E}_t(\mu_{t+1,\mathrm{L}}) + (1-\theta)\mathbb{E}(\mu_{\mathrm{R}}) + d - k\right)\\ &= \varepsilon\bar{\mu} + (1-\varepsilon)\mathbb{E}_t(\mu_{t+1,\mathrm{L}}) \qquad (17)\end{aligned}$$

where we have used the assumption that the sequence $(\mu_{t,\mathrm{R}})_{t\geq 1}$ is independent. Furthermore, by equation (16) we have

$$\mathbb{E}_t(\mu_{t+1,\mathrm{L}}) = \frac{(t+a)\mu_{t,\mathrm{L}} + \varepsilon\bar{\mu}}{t+a+\varepsilon}. \tag{18}$$

Combining equations (17) and (18) yields the conclusion. □

We omit the proof of Proposition 3.9 as the computation is similar to the above proof.

We will make use of the fact that equation (16) can be solved. The following expression is crucial for our asymptotic analysis. It expresses the current subjective mean $\mu_{t,\mathrm{L}}$ as a time-dependent weighted sum of the initial subjective mean $\mu_{0,\mathrm{L}}$ and the shocks arising from the noise traders.

Proposition 4.2. *The solution to equation* (16) *is given by*

$$\mu_{t,\mathrm{L}} = \bar{\mu} + \frac{(a)_t}{(a+\varepsilon)_t}\left(\mu_{0,\mathrm{L}} - \bar{\mu} + \frac{1-\theta}{1+r}\sum_{s=1}^{t}\frac{(a+\varepsilon)_{s-1}}{(a)_s}[\mu_{s,\mathrm{R}} - \mathbb{E}(\mu_{\mathrm{R}})]\right) \tag{19}$$

where $\bar{\mu}$ *is defined by equation* (11) *and* $(x)_n = x(x+1)\cdots(x+n-1)$ *is the Pochhammer symbol.*

Proof. The claimed solution can be verified by induction. □

Now we are ready to prove the main convergence results.

Proof of Theorems 3.3, 3.4 and 3.5. Note that

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$$

and that by Stirling's formula

$$\Gamma(s) = \sqrt{2\pi}s^{s-1/2}e^{-s}(1+O(s^{-1}))$$

as $s \to \infty$, for any constants b and c we have

$$\frac{(b)_t}{(c)_t} = \frac{\Gamma(c)}{\Gamma(b)}t^{b-c}(1+O(t^{-1})).$$

as $t \to \infty$.

In particular, we have by Proposition 4.2 the calculation

$$\mathbb{E}(\mu_{t,\mathrm{L}}) = \bar{\mu} + O(t^{-\varepsilon})$$

Similarly, upon expanding the variance of the sum by the assumed independence of the random variables $(\mu_{t,\mathrm{R}})_{t\geq 1}$, we have

$$\mathrm{Var}(\mu_{t,\mathrm{L}}) = O(t^{-2\varepsilon})\sum_{s=1}^{n} s^{2\varepsilon-2}(1+O(s^{-1}))$$

Also, by comparing the sum to the appropriate integral, we have the approximation

$$\sum_{s=1}^{t} s^p = \begin{cases} \frac{t^{p+1}}{p+1} + O(t^p) & \text{if } p > 0 \\ \frac{t^{p+1}}{p+1} + O(1) & \text{if } -1 < p \leq 0 \\ \log t + O(1) & \text{if } p = -1 \\ O(1) & \text{if } p < -1 \end{cases}$$

from which we conclude

$$\mathrm{Var}(\mu_{t,\mathrm{L}}) = O((\varsigma_t^{(\varepsilon)})^2).$$

Finally, since

$$\mathbb{E}[(\mu_{t,\mathrm{L}} - \bar{\mu})^2] = \mathrm{Var}(\mu_{t,\mathrm{L}}) + (\mathbb{E}[\mu_{t,\mathrm{L}} - \bar{\mu}])^2$$

we have proven that $\mu_{t,\mathrm{L}} \to \bar{\mu}$ in L^2. To establish almost sure convergence and the further asymptotic results, we split our work into cases.

First we suppose $0 < \varepsilon < 1/2$. Let

$$M_t = \sum_{s=1}^{t} \frac{(a+\varepsilon)_{s-1}}{(a)_s}[\mu_{s,\mathrm{R}} - \mathbb{E}(\mu_{\mathrm{R}})]. \tag{20}$$

Note that $(M_t)_{t\geq 1}$ is a martingale in its own filtration, and since it is L^2 bounded

$$\sup_{t\geq 1} \mathbb{E}[M_t^2] = \mathrm{Var}(\mu_{\mathrm{R}}) \sum_{s=1}^{\infty} \frac{(a+\varepsilon)_{s-1}^2}{(a)_s^2} < \infty.$$

The martingale convergence theorem (see Williams [15, Section 12.1]) asserts the existence of a square integrable random variable M such that

$$M_t \to M \text{ almost surely and in } L^2.$$

This proves that $\mu_{t,\mathrm{L}} \to \bar{\mu}$ almost surely, and further that

$$\Delta_t \to \frac{\Gamma(a+\varepsilon)}{\Gamma(a)} \left(\mu_{0,\mathrm{L}} - \bar{\mu} + \frac{1-\theta}{1+r} M \right) \text{ almost surely and in } L^2.$$

Since $\mathbb{E}(M) = 0$ and $\mathrm{Var}(M) = \sup_{t\geq 1} \mathbb{E}(M_t^2)$, we have established the given formulae for $\mathbb{E}(\Delta)$ and $\mathrm{Var}(\Delta)$.

Now we turn our attention to the case $1/2 \leq \varepsilon < 1$. Since the martingale $(M_t)_{t\geq 1}$ has independent increments and

$$\mathbb{E}(M_t^2) = O(t^{2\varepsilon}(\varsigma_t^{(\varepsilon)})^2)$$

we appeal to the martingale strong law of large numbers (see Williams [15, Section 12.14]) to conclude that

$$\frac{M_t}{t^{2\varepsilon}(\varsigma_t^{(\varepsilon)})^2} \to 0 \text{ almost surely.}$$

Since $t^{\varepsilon}(\varsigma_t^{(\varepsilon)})^2 \to 0$ we have $\mu_{t,\mathrm{L}} \to \bar{\mu}$ almost surely. Finally, to establish the convergence to the normal distribution, we appeal to Lindeberg's central limit theorem. (See Çinlar [7, Theorem 8.13].) □

Proof of Proposition 3.6. Recall that a sequence $(X_t)_{t\geq 1}$ converges to a random variable X in probability if $\mathbb{P}(|X_t - X| > \epsilon) \to 0$ for all $\epsilon > 0$. Convergence in probability is implied by either almost sure convergence or by L^2 convergence. Hence, it is enough to show that $(\Delta_t)_{t\geq 1}$ does not converge in probability. Note that $\limsup_{t\geq 1} \Delta_t$ is measurable with respect to the tail sigma-field. By Kolmogorov's zero-one law, the tail sigma-field is trivial, and hence $\limsup_{t\geq 1} \Delta_t$ is necessarily equal to a constant almost surely. But since the limit distribution is normal with positive variance, we must have

$$\limsup_{t\geq 1} \Delta_t = +\infty \text{ almost surely.}$$

Hence, convergence in probability is impossible. See Exercise 5.8 in the book of Chaumont & Yor [4] for further details in a very similar setting. □

Proof of Proposition 3.7. Suppose Δ is normally distributed, which implies that M is normal, where M is the limit of the martingale defined by equation (20). By Cràmer's characterisation of the normal distribution (see Feller [11, Section XV.8]) if $X + Y = M$ and if X and Y are independent then X and Y are normal. The conclusion now follows from the identity

$$M = \frac{\mu_{1,\mathrm{R}} - \mathbb{E}(\mu_{\mathrm{R}})}{a} + \sum_{s=2}^{\infty} \frac{(a+\varepsilon)_{s-1}}{(a)_s}[\mu_{s,\mathrm{R}} - \mathbb{E}(\mu_{\mathrm{R}})].$$

□

Proof of Theorem 3.8. Note that we can combine equations (19) and (8) to write the price P_t as a linear combination of the constant $\mu_{0,\mathrm{L}}$ and the random variables $(\mu_{s,\mathrm{R}})_{1\le s\le t}$. Hence the mean and autocovariance can be computed explicitly in terms of the mean $\mathbb{E}(\mu_{\mathrm{R}})$, the variance $\mathrm{Var}(\mu_{\mathrm{R}})$ and the Gamma function. The asymptotic analysis of these formulae involves Stirling's formula and the approximation of a sum by an integral just as in the proof of Theorem 3.4. The details are omitted. □

5. Conclusion

Our paper derives the rate of convergence to rational expectations equilibrium in a simple economy where the distinguished agent is a Bayesian and where the rate of convergence can be described in terms of the proportions of informed and noise traders. Furthermore, the precise mode of convergence of the distinguished agent's subjective expectations to rationality, and the standardised deviations from this limit, are established. We find that the rate of convergence is fastest when the proportion of noise traders is highest. This results suggests that the notion of an efficient market incorporating the speed of adjustment is not necessarily compatible with the idea that an efficient market contains a high proportion of informed traders.

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