FORWARD UTILITY OF INVESTMENT AND CONSUMPTION

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Abstract. Let $(S_t)_{t\geq 0}$ be a d-dimensional semimartingale market model. We define a forward utility of investment and consumption as a pair of functions U_X and U_C such that

(1) $U_X(t,\cdot,\omega)$ and $U_C(t,\cdot,\omega)$ are increasing and concave,

(2) for all $0 \leq t \leq T$ and x, we have

$$U_X(t,x) \geq \mathbb{E}\left[U_X(T, X_T^{\pi,C}) + \int_t^T U_C(s, C_s) ds | \mathcal{F}_t\right]$$

for all strategies π and C, where $X_T^{\pi,C} = x + \int_t^T \pi_s \cdot dS_s - \int_t^T C_s \; ds$, and

(3) equality holds in (2) for some pair π^* and C^*.

We characterize these functions in terms of their convex dual functions. We also show that under additional smoothness assumptions (on both the market and the utilities) such forward utility functions satisfy a certain random PDE whose solutions have an integral representation. This article is a sequel to 'A characterization of forward utility functions' by the authors and L.C.G. Rogers.

1. Introduction

There has been recent interest in the financial mathematics literature in the notion of dynamically consistent utility functions. Motivated by the classical pure-investment Merton problem, Musiela and Zariphopoulou in [11, 12, 13, 14, 15] have introduced and analyzed the idea of a forward utility function as a way to quantify the dynamically changing preferences of an investor. Independently, Henderson in [6] defined and Henderson and Hobson [7] studied a very similar object called a horizon-unbiased utility, while working in the context of finding the optimal time to sell an indivisible asset. In this paper, we broaden their definitions by introducing consumption into the story: our agent does not only invest in a financial market, but also consumes a part of her wealth at each instant.

Indeed, in the classical lifetime portfolio selection problem of Merton [10], an agent invests in a financial market and consumes part of her wealth up to an horizon T, which can be interpreted as her retirement date. The utility derived by the agent is the sum of her consumption utility between time 0 and time T and of her terminal wealth's utility. Solving the investment/consumption problem consists then in looking for the optimal investment strategy π and consumption rate C as to maximize the total expected utility

$$\mathbb{E}\Big[U_X\big(T, X_T^{\pi,C}\big) + \int_0^T U_C\big(s, C_s\big) ds\Big]$$

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where $X_T^{\pi,C}$ is the agent's wealth at time T. Merton studied this problem in the case where the assets price were modelled as geometric Brownian motions by appealing to the dynamic programming principle and solving the associated Hamilton–Jacobi–Bellman equation. In doing so, he noted that if the functions U_X and U_C are "not of the iso-elastic family, systematic effects of age will appear in the optimal decision making." See the paper of Choulli, Stricker, and Li [4] for further elaboration on this point.

However, there are problems of interest where the introduction of an horizon time T seems rather artificial. For instance, a fund manager may just aim at having her portfolio's value grow gradually as time passes, and consume a part of it (e.g. for her salary), but without having any terminal date T in mind. In such a case, it may be better to have a framework in which no horizon date T plays a particular role nor affects the problem's solution.

Indeed, if we consider our fund manager's problem, we see easily that a priori (unless the functions U_X and U_C have some particular time consistency properties), the choice of the horizon date T would affect the solution. For instance, if our fund manager was to solve the investment-consumption problem from year to year, each time fixing T one year ahead, or if she was to solve the problem by periods of two years at a time, she would probably end up taking different decisions.

The idea of a forward utility function is to take the dynamic programming equation as the *definition* of the utility functions. In this way, the optimal controls will not depend on the planning horizon by construction. To be precise, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$, and we define a forward utility of investment and consumption as a pair of functions U_X and U_C on $[0,\infty) \times \mathbb{R} \times \Omega$ such that

(1) $U_X(t,\cdot,\omega)$ and $U_C(t,\cdot,\omega)$ are increasing and concave,

(2) for all $0 \leq t \leq T$ and $\mathcal{F}_t$-measurable x, we have

$$U_X(t,x) \geq \mathbb{E}\left[U_X(T, X_T^{\pi,C}) + \int_t^T U_C(s, C_s)ds|\mathcal{F}_t\right]$$

for all admissible investment and consumption strategies π and C,

(3) equality holds in (2) for some pair π^* and C^*.

We further assume that for each $c \geq 0$ the random variable $U_C(t,c)$ is $\mathcal{F}_t$-measurable. This definition naturally extends the definitions appearing in [12] and [7]. Indeed, the original notion of forward utility corresponds to $U_C = 0$.

An example of a forward utility pair arises from the infinite horizon Merton problem. Indeed, consider a function U_C with the property that for all (t,x) there exists an investment policy π^* and consumption rate C^* such that

$$\mathbb{E}\left[\int_t^\infty U_C(s, C_s^*)ds|\mathcal{F}_t\right] \geq \mathbb{E}\left[\int_t^\infty U_C(s, C_s)ds|\mathcal{F}_t\right]$$

for all admissible strategies π, C. Letting

$$U_X(t,x) = \mathbb{E}\left[\int_t^\infty U_C(s, C_s^*)ds|\mathcal{F}_t\right],$$

it is straightforward to check that U_X, U_C is a forward utility of investment and consumption pair. However, the reverse question is more subtle: Given a utility function U_X, when does there exist a utility of consumption function U_C such that U_X, U_C is a forward utility pair? The purpose of this paper is to present a characterization of all forward utility pairs. Unsurprisingly, the key relationships will be presented in terms of convex duality.

This paper is organized as follows: In section 2 we introduce the technical assumptions on the market model and on the utility functions. In section 3 we state and prove the main result of this paper, necessary and sufficient conditions for a pair of functions to be forward utilities for a locally bounded semimartingale market. In 4, we also show, under additional smoothness assumptions on both the market and the utilities, that the convex conjugate functions satisfy a linear PDE whose solutions can be described by the integral representation of the excessive functions for space-time Brownian motion. These results extend the pure-investment forward utility characterization found in [2].

2. Set-up and notation

2.1. **Utility functions and their conjugates.**

We now introduce some assumptions on utility functions and their convex conjugates which we will need in what follows.

Assumption 2.1. *The function* $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ *satisfies*

(1) $u(x) > -\infty$ *for all* $x > 0$
(2) u *is twice-continuously differentiable, increasing and strictly concave on* $(0, \infty)$
(3) $\lim_{x \downarrow 0} u'(x) = \infty$ *and* $\lim_{x \uparrow \infty} u'(x) = 0$.

If u satisfies Assumption 2.1, then we use the notation $\hat{u}$ to denote the convex conjugate function defined by

$$\hat{u}(y) = \sup_{x>0} u(x) - xy.$$

There should be no confusion when the $\hat{}$ notation is used for functions of more than one argument, as the convex conjugation is taken with respect to the wealth variable.

Note that for all $x, y > 0$ we have Fenchel's inequality:

$$u(x) \leq \hat{u}(y) + xy.$$

The case of equality is equivalent to any of the following:

(1) $y = u'(x)$,
(2) $x = -\hat{u}'(y)$, and
(3) $u(x) = \hat{u}(y) + xy$.

Finally, we can recover u from $\hat{u}$ by

$$u(x) = \inf_{y>0} \hat{u}(y) + xy.$$

2.2. **The market.**

We consider a market with $d + 1$ traded assets, such that at least one asset has a strictly positive nominal price. By expressing all prices relative to the price of this numéraire asset, we model the (discounted) prices of the remaining d assets as a d-dimensional locally bounded càdlàg semimartingale $(S_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions with $\mathcal{F}_0$ trivial. We assume that there exists at least one probability measure locally equivalent to $\mathbb{P}$ under which the price process is a local martingale, ruling out arbitrage in the market.

The investor in this market chooses an S-integrable portfolio process π and a positive adapted càdlàg consumption rate process C so that starting from an $\mathcal{F}_t$-measurable initial wealth x at time t, the investor's wealth (in terms of the traded numéraire) at time T is given by

$$X_T^{\pi,C} = x + \int_t^T \pi_s \cdot dS_s - \int_t^T C_s \; ds$$

Since we are working with utility functions which are finite only on a half-line, we enforce the usual admissibility condition that the investor's wealth remain bounded from below by a constant.

3. A general characterization

We now present the main characterization theorem for forward utility functions. To simplify the formulae, we use the notation U' for $\frac{\partial}{\partial x}U$ and $\hat{U}'$ for $\frac{\partial}{\partial y}\hat{U}$.

Theorem 3.1. *Suppose that $U_X(t,\cdot,\omega)$ and $U_C(t,\cdot,\omega)$ satisfies Assumption 2.1 for all (t,ω). Then the following are equivalent:*

1. *U_X, U_C is a forward utility pair.*
2. *For all $\mathcal{F}_t$-measurable $y>0$ there exists an equivalent martingale density Z such that for all $0 \le t \le T$ the following hold true:*
   - $\hat{U}_X(t,y) = \mathbb{E}\left[\hat{U}_X\left(T, y\frac{Z_T}{Z_t}\right) + \int_t^T \hat{U}_C\left(u, y\frac{Z_u}{Z_t}\right) du | \mathcal{F}_t\right]$
   - $\hat{U}'_X(t,y) = \mathbb{E}\left[\frac{Z_T}{Z_t}\hat{U}'_X\left(T, y\frac{Z_T}{Z_t}\right) + \int_t^T \frac{Z_u}{Z_t}\hat{U}'_C\left(u, y\frac{Z_u}{Z_t}\right) du | \mathcal{F}_t\right]$
   - $-\hat{U}'_X\left(T, y\frac{Z_T}{Z_t}\right) - \int_t^T \hat{U}'_C\left(u, y\frac{Z_u}{Z_t}\right) du$ *is attainable from* $-\hat{U}'_X(t,y)$.

Remark 1. Since the techniques of convex duality are well-developed in analyzing utility maximization problems in finance, the appearance of the dual functions $\hat{U}_X$ and $\hat{U}_C$ should come as no surprise. Schachermayer [16] introduced of the dynamic form of the dual problem in the anaylsis of the supermartingale property of the optimal wealth process in a locally bounded semimartingale market with utility finite on the entire real line. Biagini and Fritelli [3] extended this analysis to the not-locally-bouned case, showing that the optimal wealth is attainable if the min-max σ-martingale measure is equivalent to $\mathbb{P}$.

We begin with a little lemma that shows that the optimal controls do not depend on the horizon as claimed in Section 1. This fact demands a proof as our definition of forward utility seems to leave open the possibility that the optimal portfolio and consumption rate may depend on T.

Lemma 3.2. *Let U_X, U_C be a forward utility pair. Then, for any fixed $t \ge 0$ and $\mathcal{F}_t$-measurable $x>0$, there exists admissible trading strategy $(\pi^*_u)_{u\ge t}$ and consumption process $(C^*_u)_{u\ge t}$ such that*

$$U_X(u, X^*_u) = \mathbb{E}\left[U_X(T, X^*_T) + \int_u^T U_C(s, C^*_s)\; ds | \mathcal{F}_u\right]$$

for all $t \le u \le T$, where $X^*_u = X_u^{\pi^*,C^*}$.

Proof Lemma 3.2. Fix $0 \le t \le u \le T$ and $\mathcal{F}_t$-measurable initial wealth $x>0$. By assumption, there exists admissible strategy π^*, C^*, possibly depending on T, such that

$$U_X(t,x) = \mathbb{E}\left[U_X(T, X^*_T) + \int_t^T U_C(s, C^*_s) ds | \mathcal{F}_t\right]$$

Notice by the definition of forward utility

$$(*)\qquad \mathbb{E}\left[U_X(T,X_T^*)+\int_u^T U_C(s,C_s^*)ds|\mathcal{F}_u\right]\le U_X(u,X_u^*).$$

Applying the tower property of conditional expectations we have

$$\begin{aligned}
U_X(t,x) &= \mathbb{E}\left[U_X(T,X_T^*)+\int_t^T U_C(s,C_s^*)ds|\mathcal{F}_t\right]\\
&= \mathbb{E}\left\{\mathbb{E}\left[U_X(T,X_T^*)+\int_u^T U_C(s,C_s^*)ds|\mathcal{F}_u\right]+\int_t^u U_C(s,C_s^*)ds|\mathcal{F}_t\right\}\\
&\le \mathbb{E}\left[U_X(u,X_u^*)+\int_t^u U_C(s,C_s^*)ds|\mathcal{F}_t\right].
\end{aligned}$$

Again by the definition of forward utility, the inequality above can only be an almost sure equality and we see that the controls π^*, C^* are optimal over the subinterval $[t,u]$. Furthermore, inequality (*) must hold with equality almost surely, proving the claim. □

Proof of Theorem 3.1. (2) $\Rightarrow$ (1). We begin with the easier direction. Fix $t\ge 0$ and a $\mathcal{F}_t$-measurable $x>0$. Let $y=U_X'(t,x)$ and let $(Z_t)_{t\ge 0}$ be the density of the corresponding martingale measure. If $X_T=x+\int_t^T\pi_s\cdot dS_s-\int_t^T C_s ds$ is attainable from x then the following inequality holds

$$\begin{aligned}
U_X(T,X_T)+\int_t^T U_C(s,C_s)ds &\le \hat{U}_X\left(T,y\frac{Z_T}{Z_t}\right)+y\frac{Z_T}{Z_t}X_T+\int_t^T\left(\hat{U}_C\left(s,y\frac{Z_s}{Z_t}\right)+y\frac{Z_s}{Z_t}C_s\right)ds\\
&= \hat{U}_X\left(T,y\frac{Z_T}{Z_t}\right)+\int_t^T \hat{U}_C\left(s,y\frac{Z_s}{Z_t}\right)ds\\
&\quad +y\frac{Z_T}{Z_t}\left(x+\int_t^T\pi_s\cdot dS_s\right)+\int_t^T y\frac{(Z_s-Z_T)}{Z_t}C_s ds
\end{aligned}$$

by the definition of convex conjugate. Since the portfolio process is admissible, the stochastic integral is bounded from below, and hence is a supermartingale for the martingale measure with density process Z. Taking conditional expectations yields

$$\begin{aligned}
\mathbb{E}\left[U_X(T,X_T)+\int_t^T U_C(s,C_s)ds|\mathcal{F}_t\right] &\le \mathbb{E}\left[\hat{U}_X\left(T,y\frac{Z_T}{Z_t}\right)+\int_t^T \hat{U}_C\left(s,y\frac{Z_s}{Z_t}\right)ds|\mathcal{F}_t\right]+xy\\
&= \hat{U}_X(t,y)+xy\\
&= U_X(t,y)
\end{aligned}$$

where the last line follows from the choice of y.

Now by choosing the consumption rate as

$$C_s^*=-\hat{U}_C'\left(s,y\frac{Z_s}{Z_t}\right)$$

and the portfolio strategy by

$$x+\int_t^T\pi_s^*\cdot dS_s=-\hat{U}_X'\left(T,y\frac{Z_T}{Z_t}\right)+\int_t^T C_s^*\,ds,$$

which is attainable by assumption, we see that we have equality in the above inequalities. This completes the proof that U_X, U_C is a forward utility pair.

(1) $\Rightarrow$ (2). Suppose U_X, U_C is a forward utility pair. Fix $t \geq 0$ and $\mathcal{F}_t$-measurable $y > 0$, let $x = -\hat{U}'_X(t, y)$, and let $(X^*_u, C^*_u)_{u \geq t}$ be the optimal wealth and consumption process with $X^*_t = x$. Let $(Z_s)_{s \in [0,t]}$ be the density of any equivalent martingale measure on the interval $[0, t]$, and let

$$Z_u = \frac{U'_X(u, X^*_u) Z_t}{y}$$

for $u \geq t$.

Now, we must show that $(Z_u)_{u \geq t}$ is the density of an equivalent martingale measure. Fix $t \leq u < T$ and let $(\pi_s)_{s \in [u,T]}$ be an admissible trading strategy such that $(X_s)_{s \in [u,T]}$ is bounded, where $X_s = X_u + \int_u^s \pi_\tau \cdot dS_\tau$ and $X_u > 0$. Then $\epsilon X_T + X^*_T$ is attainable from $\epsilon X_u + X^*_u$ for all $\epsilon \in \mathbb{R}$. Hence

$$\mathbb{E}[U_X(T, \epsilon X_T + X^*_T) - U_X(T, X^*_T) | \mathcal{F}_u] \leq U_X(u, \epsilon X_u + X^*_u) - U_X(u, X^*_u).$$

Dividing by ϵ and letting $\epsilon \downarrow 0$ yields

$$\mathbb{E}[U'_X(T, X^*_T) X_T | \mathcal{F}_u] \leq U'_X(u, X^*_u) X_u$$

by the monotone convergence theorem and the concavity of U. Similarly, letting $\epsilon \uparrow 0$ yields the reverse inequality, so that by linearity

$$\mathbb{E}[Z_T X_T | \mathcal{F}_u] = Z_u X_u$$

for all bounded X_T attainable from X_u. In particular, letting the wealth $X_u = X_T = 1$ shows that $(Z_u)_{u \geq t}$ is a martingale. Similarly, letting $X_s = S^i_{s \wedge \tau_N}$ for some $i \in \{1, \ldots, d\}$ where $\tau_N = \inf\{s \geq u : |S_s| \geq N\}$ shows that the equivalent measure $\mathbb{Q}$ induced by the martingale Z is such that the discounted prices are local martingales.

Now, writing $X^*_s = \xi_s - \int_t^s C^*_\tau d\tau$ where $\xi_s = x + \int_t^s \pi^*_\tau \cdot dS_\tau$ is bounded from below, we can repeat the above argument by taking $X_s = \epsilon \xi_s$. Note that we have dropped the assumption that the wealth process is bounded since $(1+\epsilon)\xi_T$ is attainable from $(1+\epsilon)\xi_u$ for any $\epsilon > -1$. We can conclude that $(\xi_u)_{u \geq t}$ is a martingale under $\mathbb{Q}$.

Now let $(C_u)_{u \geq t}$ be a bounded càdlàg process. Again, by optimality, we have

$$\mathbb{E}\left\{ U_X\left(T, X^*_T - \epsilon \int_t^T C_u du\right) - U_X(T, X^*_T) + \int_t^T [U_C(u, C^*_u + \epsilon C_u) - U_C(u, C^*_u)] du | \mathcal{F}_t \right\} \leq 0$$

By dividing by ϵ and then taking $\epsilon \downarrow 0$ (and then $\epsilon \downarrow 0$) we have by the same argument as before:

$$\mathbb{E}\left\{ \int_t^T [U'_X(T, X^*_T) - U'_C(u, C^*_u)] C_u du | \mathcal{F}_t \right\} = 0.$$

Since the perturbation was arbitrary, we have

$$U'_C(u, C^*_u) = \mathbb{E}[U'_X(T, X^*_T) | \mathcal{F}_u] = y \frac{Z_u}{Z_t}$$

for Lebesgue almost all $u \in [t, T]$. In particular, $C^*_u = -\hat{U}'_C\left(u, y\frac{Z_u}{Z_t}\right)$. Therefore, we can conclude that

$$-\hat{U}'_X\left(T, y\frac{Z_T}{Z_t}\right) - \int_t^T \hat{U}'_C\left(u, y\frac{Z_u}{Z_t}\right) du = \xi_T$$

is attainable from its conditional mean $\mathbb{E}^{\mathbb{Q}}[\xi_T|\mathcal{F}_t] = x$.

Finally, we have

$$\begin{aligned}
\mathbb{E}\left[\hat{U}_X\left(T, y\frac{Z_T}{Z_t}\right) + \int_t^T \hat{U}_C\left(u, y\frac{Z_u}{Z_t}\right) du|\mathcal{F}_t\right] &= \mathbb{E} + \Bigg\{U_X\left(T, X_T^*\right) - y\frac{Z_T}{Z_t}X_T^* \\
&\quad \int_t^T \left[U_C\left(u, C_u^*\right) du - y\frac{Z_u}{Z_t}C_T^*\right] |\mathcal{F}_t\Bigg\} \\
&= U_X(t, X_t^*) - yX_t^* \\
&= \hat{U}_X\left(t, y\right).
\end{aligned}$$

□

We state the following corollary, which easily follows from the Fenchel inequality:

Corollary 3.3. *If U_X, U_C is a forward utility pair then*

$$\hat{U}_X(t, y) \leq \mathbb{E}\left[\hat{U}_X\left(T, y\frac{Z_T}{Z_t}\right) + \int_t^T \hat{U}_C\left(u, y\frac{Z_u}{Z_t}\right) du|\mathcal{F}_t\right]$$

for all equivalent martingale measure densities Z.

From this theorem, we may construct a family of examples such that $U_X(0, x, \omega) = u(x)$ for a given (nonrandom) utility function u.

Corollary 3.4. *Let u satisfy Assumption 2.1. Assume there exists an equivalent martingale measure with density process $(Z_t^*)_{t\geq 0}$ such that $1/Z_T^*$ is attainable for all $T \geq 0$. Then the functions U_X, U_C defined by*

$$U_X(t, x, \omega) = \bar{F}(t)u\left(\frac{xZ_t^*(\omega)}{\bar{F}(t)}\right), \quad U_C(t, c, \omega) = f(t)u\left(\frac{cZ_t^*(\omega)}{f(t)}\right),$$

is a forward utility pair where $f > 0$ is a probability density on $[0, \infty)$ and $\bar{F}(t) = \int_t^\infty f(s)ds$ is its complementary distribution function.

Remark 2. The wealth process $(X_t^*)_{t\geq 0}$ above is called the numéraire portfolio process and was characterized by Becherer in [1] as the optimizer of the standard Merton pure investment problem under logarithmic utility. In the continuous price setting, the equivalent martingale measure corresponding to the martingale $(Z_t^*)_{t\geq 0}$ is called the Föllmer–Schweizer minimal martingale measure. The striking feature of Corollary 3.4 is that the optimal investment behavior for an agent using this forward utility with arbitrary initial indirect utility u is exactly the same as for an agent maximizing expected logarithmic utility.

4. Smoothness in time

In this section, we explore the consequence of an additional assumption that $t \mapsto U(t, x, \omega)$ is smooth in time. Just in the case of the pure problem studied in [2], this assumption greatly restricts the possibilities, though not nearly to the same extent.

4.1. **Motivation.** To motivate this study, we note that the strategy of investing no money $\pi = 0$ in the risky assets and consuming a constant amount $c > 0$ is admissible for each intial wealth $x > 0$, we have

$$\mathbb{E}[U_X(T,x) + \int_t^T U_C(s,c)ds|\mathcal{F}_t] \leq U_X(t,x).$$

If the forward utility of consumption U_C is a non-negative process, then the process $(U_X(t,x))_{t\geq 0}$ is a super-martingale. That is, the forward utility of wealth decreases on average. If we assume that $t \mapsto U_X(t,x)$ is differentiable, then the utility must, in fact, decrease almost surely. Models in this section, therefore, have the feature that the investor is worse off *with probability one* by not investing in the market.

4.2. **Further assumptions on the market model.** For the rest of this section, we restrict the class of market models from those with locally bounded prices to those with continuous prices. We will assume that the prices have the decomposition

$$dS_t^i = S_t^i \left(\mu_t^i \, dt + \sum_{j=1}^d \sigma_t^{i,j} \, dW_t^j \right)$$

for $i \in \{1, \ldots, d\}$ and previsible processes $(\mu_t)_{t\geq 0}$ and $(\sigma_t)_{t\geq 0}$ and k-dimensional Brownian motion $W = (W_t)_{t\geq 0}$. We assume that the Brownian motion is $(\mathcal{F}_t)_{t\geq 0}$-adapted, but we do not necessarily assume that the market is complete, i.e. $(\mathcal{F}_t)_{t\geq 0}$ may be strictly larger than $(\mathcal{F}_t^W)_{t\geq 0}$, the filtration generated by W.

For each (t,ω) we let $\Theta_t(\omega)$ be the k-dimensional vector of smallest Euclidean norm such that

$$\sigma_t(\omega)\Theta_t(\omega) = \mu_t(\omega).$$

Note that the vector $\Theta_t(\omega)$ is in the orthogonal complement of the kernel of the matrix $\sigma_t(\omega)$; in other words, $\Theta_t(\omega)$ is in the range of the transpose matrix $\sigma_t(\omega)^{\mathrm{T}}$. We let $(\sigma_t^{\mathrm{T}})^{-1}\Theta_t$ denote the solution λ of $\sigma_t^{\mathrm{T}}\lambda = \Theta_t$, though we do not necessarily assume that σ_t is invertible.

We let

$$Z_t^* = \exp\left(-\int_0^t \Theta_u \cdot dW_u - \frac{1}{2}\int_0^t |\Theta_u|^2 du \right)$$

and assume that $\mathbb{E}(Z_t^*) = 1$ for all $t \geq 0$ so that $(Z^*)_{t\geq 0}$ is the density process of the minimal martingale measure. Of course, because of the possibility of incompleteness, there may exist plenty of other equivalent martingale measures.

To prove the main result of this section, we must make the following assumptions on the market.

Assumption 4.1. *The market price of risk process $(\Theta_t)_{t\geq 0}$ is continuous, bounded uniformly in $(t,\omega) \in [0,\infty) \times \Omega$, and satisfies the non-degeneracy conditions*

(1) $\Theta_t \neq 0$ *almost surely for all* $t \geq 0$, *and*
(2) $\int_0^\infty |\Theta_u|^2 \, du = \infty$ *almost surely.*

Assumption 4.1 will be in force throughout the remainder of this section.

In what follows, we will use the notation

$$A_t = \int_0^t |\Theta_s|^2 \, ds$$

to denote the mean-variance tradeoff process. We will also use the notation $\mathcal{E}(\cdot)$ for the Dooléans–Dade stochastic exponential, so that, for instance,

$$Z_t^* = \mathcal{E}\left(-\int_0^t \Theta_u \cdot dW_u\right).$$

4.3. **The main results.** We are now ready to state and prove the main result of this section. We will need an extra assumption.

Assumption 4.2. (1) *The functions $U_X(t,\cdot,\omega)$ and $U_C(t,\cdot,\omega)$ satisfy Assumption 2.1, for all $t \geq 0$ and almost all $\omega \in \Omega$.*

(2) *The function U_C and the partial derivatives $\frac{\partial^2 U_X}{\partial t \partial x} = \frac{\partial^2 U_X}{\partial x \partial t}$, $\frac{\partial^3 U_X}{\partial x^3}$, and $\frac{\partial U_C}{\partial x}$ are continuous on $[0,\infty) \times (0,\infty)$ almost surely.*

(3) *For each $T > 0$, there are constants $C > 0$ and $k > 0$ such that*

$$\limsup_{x \uparrow \infty} x^k \frac{\partial U_X}{\partial x} \leq C$$

for all $t \in [0,T]$ almost surely.

Theorem 4.3. *Under Assumption 4.2 the following are equivalent:*

(1) *U_X, U_C is a forward utility pair*

(2) $\dfrac{\partial U_X}{\partial t} - \dfrac{|\Theta_t|^2}{2} \dfrac{\left(\frac{\partial U_X}{\partial x}\right)^2}{\frac{\partial^2 U_X}{\partial x^2}} + \hat{U}_C\left(t, \dfrac{\partial U_X}{\partial x}\right) = 0$

(3) $\dfrac{\partial \hat{U}_X}{\partial t} + \dfrac{|\Theta_t|^2}{2} y^2 \dfrac{\partial^2 \hat{U}_X}{\partial y^2} + \hat{U}_C = 0$

Proof of Theorem 4.3. Since the equivalence of (2) and (3) is easily verified under the assumption of the Inada condition by the chain rule of calculus, we need only show (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Fix $t > 0$ and an $\mathcal{F}_t$-measurable $x > 0$. Recall that the variation of our agent's wealth, taking into account the budget and self-financing constraints, is equal to

$$dX_u^{\pi,C} = \sigma_u^{\mathrm{T}} \Pi_u \cdot \Big[\Theta_u du + dW_u\Big] - C_u \, du$$

where we are now using the notation $\Pi_u = \mathrm{diag}(S_u)\pi_u$.

By the assumed differentiability in time, the semimartingale $(U_X(t,x))_{t\geq 0}$ has no local-martingale part. Hence, by the generalized Itô formula (see Theorem 3.3.1 of Kunita's book [9], for instance), we have the decomposition

$$\begin{aligned} U_X(T, X_T^{\pi,C}) &= U_X(t,x) + \int_t^T \frac{\partial U_X}{\partial x} \sigma_u^{\mathrm{T}} \Pi_u \cdot dW_u \\ &\quad + \int_t^T \left(\frac{\partial U_X}{\partial t} + \frac{\partial U_X}{\partial x}(\sigma_u^{\mathrm{T}} \Pi_u \cdot \Theta_u - C_u) + \frac{1}{2}\frac{\partial^2 U_X}{\partial x^2} |\sigma_u^{\mathrm{T}} \Pi_u|^2 \right) du. \end{aligned}$$

where the partial derivatives in the integral are evaluated at $(u, X_u^{\pi,C})$.

Since the processes $\left(U_X(u, X_u^{\pi,C}) + \int_t^u U_C(s, C_s) ds\right)_{u \geq t}$ is a super-martingale for all admissible controls π, C we have

$$(1) \qquad \int_t^T \left(\frac{\partial U_X}{\partial t} + \frac{\partial U_X}{\partial x} \sigma_u^{\mathrm{T}} \Pi_u \cdot \Theta_u + \frac{1}{2}\frac{\partial^2 U_X}{\partial x^2} |\sigma_u^{\mathrm{T}} \Pi_u|^2 + U_C - C_u \frac{\partial U_X}{\partial x} \right) du \leq 0,$$

with equality for the optimal controls π^*, C^*.

Let π^*, C^* be an optimal strategy as in Lemma 3.2, and let $X^*_u = X^{\pi^*,C^*}_u$ be the associated optimal discounted wealth. By the concavity of U_X and Fenchel's inequality we have

$$\begin{aligned}\int_t^T \left(\frac{\partial U_X}{\partial t} - \frac{1}{2}|\Theta_u|^2 \frac{\left(\frac{\partial U_X}{\partial x}\right)^2}{\frac{\partial^2 U_X}{\partial x^2}} + \hat{U}_C \right) du &= \frac{1}{2}\int_t^T \frac{-1}{\frac{\partial^2 U_X}{\partial x^2}} \left| \frac{\partial U_X}{\partial x}\Theta_u + \frac{\partial^2 U_X}{\partial x^2}\sigma_u^{\mathrm{T}}\Pi^*_u \right|^2 du \\ &\quad + \int_t^T \left(\hat{U}_C - U_C + C^*_u \frac{\partial U_X}{\partial x} \right) du \\ &\geq 0\end{aligned}$$

where the partial derivatives of U_X are evaluated at (u, X^*_u) and $\hat{U}_C$ is evaluated at $(u, \frac{\partial U_X}{\partial x})$.

Since the wealth process is continuous and the consumption is right-continuous, and the partial derivatives of the utility functions are continuous, we may take the limit as $T \downarrow t$ to conclude that

$$\frac{\partial U_X}{\partial t} \geq \frac{1}{2}|\Theta_t|^2 \frac{\left(\frac{\partial U_X}{\partial x}\right)^2}{\frac{\partial^2 U_X}{\partial x^2}} - \hat{U}_C$$

almost surely for all $x > 0$ and $t \geq 0$.

We now define a locally optimal controls π^+ and C^+ by taking, for times u between t and T:

$$\pi^+_u = -\mathbb{1}_{\{X^+_u \geq 0\}} \frac{\frac{\partial U_X}{\partial x}}{\frac{\partial^2 U_X}{\partial x^2}}(t,x)\,(\sigma_u^{\mathrm{T}})^{-1}\Theta_u$$

and

$$C^+_u = -\mathbb{1}_{\{X^+_u \geq 0\}} \frac{\partial \hat{U}_C}{\partial y}\left(t, \frac{\partial U_X}{\partial x}(t,x)\right)$$

with corresponding wealth $X^+_u = X^{\pi^+,C^+}_u$. This investment/consumption strategy is admissible thanks to the indicator function.

By equation (1) we have

$$\begin{aligned}0 \;\geq\; &\int_t^T \left[\frac{\partial U_X}{\partial t}(u, X^+_u) - \frac{|\Theta_u|^2}{2} \frac{\left(\frac{\partial U_X}{\partial x}\right)^2}{\frac{\partial^2 U_X}{\partial x^2}}(t,x)\mathbb{1}_{\{X^+_u \geq 0\}} \left(2\frac{\frac{\partial U_X}{\partial x}(u, X^+_u)}{\frac{\partial U_X}{\partial x}(t,x)} - \frac{\frac{\partial^2 U_X}{\partial x^2}(u, X^+_u)}{\frac{\partial^2 U_X}{\partial x^2}(t,x)} \right) \right] du \\ &+ \int_t^T \left(U_C(u, C^+_u) - C^+_u \frac{\partial U_X}{\partial x}(t, X^+_u) \right) du\end{aligned}$$

Again, taking the limit $T \downarrow t$ we conclude that

$$\frac{\partial U_X}{\partial t} \leq \frac{1}{2}|\Theta_t|^2 \frac{\left(\frac{\partial U_X}{\partial x}\right)^2}{\frac{\partial^2 U_X}{\partial x^2}} - \hat{U}_C,$$

completing the proof.

(3) $\Rightarrow$ (1) We need only check that the three conditions appearing in Theorem 3.1 hold. But by the generalized Itô formula, we have

$$\hat{U}_X(t,y) = \hat{U}_X\left(T, y\frac{Z^*_T}{Z^*_t}\right) + \int_t^T \hat{U}_C\left(u, y\frac{Z^*_u}{Z^*_t}\right) du + \int_t^T \frac{\partial \hat{U}_X}{\partial y}\left(u, y\frac{Z^*_u}{Z^*_t}\right) y\frac{Z^*_u}{Z^*_t}\Theta_u \cdot dW_u$$

By the assumption that the process Θ is bounded and the assumed growth bound on the utility functions (see Ekeland and Taflin [5]), the above stochastic integral is a martingale and the first condition of Theorem 3.1 follows on taking conditional expectations on both sides. Now, the convexity and the assumed differentiability of $\hat{U}_X(t,.)$ and $\hat{U}_C(t,.)$ allow us to differentiate the first condition of Theorem 3.1 with respect to y, and applying the monotone convergence theorem yields the second condition.

Finally, applying again the generalized Itô formula, we get

$$\begin{aligned}\frac{\partial \hat{U}_X}{\partial y}(t,y) &= \frac{\partial \hat{U}_X}{\partial y}\left(T, y\frac{Z_T^*}{Z_t^*}\right) + \int_t^T \frac{\partial \hat{U}_C}{\partial y}\left(u, y\frac{Z_u^*}{Z_t^*}\right) du \\ &\quad + \int_t^T \frac{\partial^2 \hat{U}_X}{\partial y^2}\left(u, y\frac{Z_u^*}{Z_t^*}\right) y\frac{Z_u^*}{Z_t^*}\Theta_u \cdot (dW_u + \Theta_u \, du).\end{aligned}$$

proving the attainability condition of Theorem 3.1. □

Remark 3. From this theorem, we can construct many examples of forward utility pairs. For instance, one can check that the pair

$$U_X(t,x) = \frac{1}{1-R}(x^{1-R}e^{\frac{1}{2}(1-R^{-1})A_t} - 1)e^{-QA_t}$$

and

$$U_C(t,x) = \frac{1}{1-R}(x^{1-R}e^{\frac{1}{2}(1-R^{-1})A_t} - 1)Q|\Theta_t|^2 e^{-QA_t}$$

satisfy the nonlinear primal PDE, hence are a forward utility pair for $R > 0$ and $Q > 0$. Since the dual PDE is linear, one can build futher examples by convex combinations of the above solutions. Indeed, if

$$\hat{U}_X(t,y) = \int_{(0,\infty)\times[0,\infty)} \frac{1}{1-r}(1 - e^{\frac{1}{2}r(1-r)A_t}y^{1-r})e^{-qA_t}\nu(dr,dq)$$

and

$$\hat{U}_C(t,y) = \int_{(0,\infty)\times[0,\infty)} \frac{1}{1-r}(1 - e^{\frac{1}{2}r(1-r)A_t}y^{1-r})q|\Theta_t|^2 e^{-qA_t}\nu(dr,dq)$$

for a finite measure ν with compact support, then U_X, U_C is a forward utility pair.

How much further can the idea in the above remark be taken? By making the substitution $z = \log y - \frac{1}{2}A_t$ and letting $H(t,z) = -\frac{\partial}{\partial y}\hat{U}(t, ze^{A_t/2})$ we have

$$\frac{\partial H_X}{\partial t} + \frac{1}{2}|\Theta_t|^2 \frac{\partial^2 H_X}{\partial z^2} = -H_C.$$

Since $y \mapsto \hat{U}(t,y)$ is decreasing, we see that H_X is positive and

$$\frac{\partial H_X}{\partial t} + \frac{1}{2}|\Theta_t|^2 \frac{\partial^2 H_X}{\partial z^2} \le 0.$$

That is to say, H_X is, up to scaling, an excessive function for space-time Brownian motion. Such functions have an integral representation in terms of extremal excessive functions. In the case of space-time Brownian motion, these functions are known explicitly in terms of the

Brownian transition kernel; see the chapter of Sieveking [17] from 1968 or the more recent article of Janssen [8]. In particular, we find that

$$\begin{aligned}\hat{U}_X(t,y) &= \int_{u=t}^{\infty}\int_{z=-\infty}^{\infty} \hat{U}_C(u, ye^{-\frac{1}{2}(A_u-A_t)+\sqrt{A_u-A_t}z})\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz\ du \\ &\quad + \int_{(0,\infty)} \frac{1-e^{\frac{1}{2}r(1-r)A_t}y^{1-r}}{1-r}\nu(dr)\end{aligned}$$

for a finite measure ν. Note that the second term appeared in [2] in the integral representation of pure investment forward utilities. Just as an excessive function has a Riesz decomposition as the sum of a potential and a harmonic function, we can interpret the above representation to say that the convex conjugate of a forward utility of wealth is the sum of the conjugates of a forward utility arising from an infinite horizon optimal consumption problem and a pure investment problem. However, note that the left-hand side of the above equation is $\mathcal{F}_t$-measurable, implying that the right-hand side cannot be chosen arbitrarily. One choice that satisfies the measurability conditions is given below. Let V_X and V_C be two deterministic functions on $[0,\infty)\times\mathbb{R}$ satisfying the dual Inada conditions and

$$V_X(t,y) = \int_{h=0}^{\infty}\int_{z=-\infty}^{\infty} V_C(t+h, ye^{-\frac{1}{2}h+\sqrt{h}z})\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz\ dh$$

Then U_X, U_C is a forward utility pair where

$$\hat{U}_X(t,y,\omega) = V_X(A_t(\omega), y) \quad \text{and} \quad \hat{U}_C(t,y,\omega) = |\Theta_t(\omega)|^2 V_C(A_t(\omega), y)$$

Note that we have

$$\hat{U}_X(t,y) = \mathbb{E}\left[\int_t^{\infty} \hat{U}_C\left(u, y\frac{Z_u^*}{Z_t^*}\right) du|\mathcal{F}_t\right]$$

if the increasing process A and the Brownian motion W are independent, but the above formula fails to hold in general.

From the proof of Theorem 4.3 we have the following 'mutual fund theorem':

Corollary 4.4. *If U_X, U_X is a forward utility pair satisfying Assumption 4.2, then for every $(t,x)\in[0,\infty)\times(0,\infty)$ the optimal strategy π^* satisfies*

$$\Pi_u^* = c_u(\sigma_u^{\mathrm{T}})^{-1}\Theta_u$$

for $u \ge t$, where c_u is a positive scalar-valued random variable.

Finally we can refine Corollary 3.3 in the case of forward utilities satisfying the additional smoothness assumption.

Corollary 4.5. *If U_X, U_X is a forward utility pair satisfying Assumption 4.2, then the convex conjugate function $\hat{U}$ satisfies*

$$\hat{U}_X(t,y) = \mathbb{E}\left[\hat{U}_X\left(T, y\frac{Z_T^*}{Z_t^*}\right) + \int_t^T \hat{U}_C\left(T, y\frac{Z_u^*}{Z_t^*}\right) du\Big|\mathcal{F}_t\right].$$

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References

[1] Becherer, D.: The numéraire portfolio for unbounded semimartingales. Finance and Stochastics **5**, 327-341 (2001)

[2] Berrier, F.P.Y.S, L.C.G. Rogers, and M.R. Tehranchi: A characterization of forward utilities. Preprint (2008)

[3] Biagini, S. and M. Fritelli: The supermartingale property of the optimal wealth process for general semimartingales. Finance and Stochastics **11**(2), 253–266 (2007)

[4] Choulli, T., C. Stricker, and J. Li: Minimal Hellinger martingale measures of order q.Finance and Stochastics **11**(3), 1432–1122 (2007)

[5] Ekeland, I., and E. Taflin: A theory of bond portfolios. Annals of Applied Probability **15** (2), 1260 – 1305 (2005)

[6] Henderson, V.: Valuing the option to invest in an incomplete market. Mathematics and Financial Economics **1**(2) 103–128 (2007)

[7] Henderson, V. and D. Hobson: Horizon-unbiased utility functions. Stochastic Processes and their Applications **117**(11), 1621–1641 (2007)

[8] Janssen, K.: Integral representation for space-time excessive functions. In Potential Theory in Matsue, H. Aikawa (eds.) Advanced Studies in Pure Mathematics **44**, 167–177 (2006)

[9] Kunita, H.: Stochastic Flows and Stochastic Differential Equations, Cambridge University Press (1997)

[10] Merton, R.: Lifetime portfolio selection under uncertainty: the continuous-time case. Review of Economic Statistics **51**, 247–257 (1969)

[11] Musiela, M. and Th. Zariphopoulou: Backward and forward utilities and the associated indifference pricing systems: The case study of the binomial model. In R. Carmona (ed.): Indifference Pricing, Princeton University Press (2005)

[12] Musiela, M. and Th. Zariphopoulou. Investment and forward utilities. Preprint (2006)

[13] Musiela, M. and Th. Zariphopoulou. Investment performance measurement, risk tolerance and optimal portfolio choice. Preprint (2007)

[14] Musiela, M. and Th. Zariphopoulou: Implied preferences and bespoke portfolios. Preprint (2007)

[15] Musiela, M. and Th. Zariphopoulou: Stochastic partial differential equations and portfolio choice. Preprint (2007)

[16] Schachermayer, W.: A super-martingale property of the optimal portfolio process. Finance and Stochastics **7**(4),433–456 (2003)

[17] Sieveking, M.: Integraldarstellung superharmonischer Funktionen mit Anwendung auf parabolische Differentialgleichungen. In Lecture Notes in Mathematics **69**, Springer (1968)

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