A CHARACTERIZATION OF FORWARD UTILITY FUNCTIONS

F.P.Y.S. BERRIER, L.C.G. ROGERS, M.R. TEHRANCHI*

Abstract. Recently, a notion of dynamically consistent utility functions has appeared in the mathematical finance literature. In this paper, we call a function $U : [0,\infty) \times \mathbb{R} \times \Omega \to \mathbb{R} \cup \{-\infty\}$ a forward utility if

(1) $x \mapsto U(t, x, \omega)$ is increasing and concave,
(2) $U(t, x) \geq \mathbb{E}[U(T, X)|\mathcal{F}_t]$ for all attainable X, and
(3) there exists an attainable X^* such that $U(t, x) = \mathbb{E}[U(T, X^*)|\mathcal{F}_t]$

Working in a fairly general semimartingale market, we present a complete characterization of the forward utility functions in terms of their convex conjugate functions. In the case when the forward utility is further assumed to be decreasing in time and the asset prices continuous (though the market may be incomplete), we present an explicit integral representation of their convex conjugate functions. As a corollary, we prove that a function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ can be extended to a decreasing forward utility U with $U(0, x, \omega) = u(x)$ if and only if $z \mapsto (u')^{-1}(e^z)$ is the Laplace transform of a finite measure.

1. Introduction

Consider a market with $d+1$ assets. We assume that one asset has a strictly positive price for all time, and that the prices of the other d assets, when discounted by this numéraire asset, is modelled as a d-dimensional semimartingale $(S_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and such that $\mathcal{F}_0$ is trivial. For a given $t \geq 0$, a positive $\mathcal{F}_t$-measurable random variable X_t will be called a t-wealth, as such random variables will model the wealth of an investor in this market. Given a t-wealth X_t, an admissible trading strategy is a d-dimensional predictable S-integrable process $(H_u)_{u>t}$ such that

$$X_t + \int_t^T H_u \cdot dS_u > 0 \text{ almost surely for all } T \geq t.$$

We say a T-wealth X_T is attainable from the t-wealth X_t if there exists an admissible trading strategy $(H_u)_{u\in(t,T]}$ such that $X_T = X_t + \int_t^T H_u \cdot dS_u$. Let $\mathcal{Z}$ be the set of positive supermartingales $(Z_u)_{u\geq 0}$ such that $Z_0 = 1$ and

$$\mathbb{E}(Z_T X_T|\mathcal{F}_t) \leq Z_t X_t \text{ almost surely}$$

for every $0 \leq t \leq T$ and every T-wealth X_T attainable from the t-wealth X_t. We assume that the set $\mathcal{Z}$ is not empty, implying that the market model is arbitrage-free.

We call a function $U : [0,\infty) \times \mathbb{R} \times \Omega \to \mathbb{R} \cup \{-\infty\}$ a random utility function iff

(1) $x \mapsto U(t, x, \omega)$ is increasing, concave, and finite valued on $(0,\infty)$ for all $(t, \omega) \in [0,\infty) \times \Omega$,

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* Corresponding author. PRELIMINARY VERSION: Please do not circulate.

(2) $\omega \to U(t,x,\omega)$ is $\mathcal{F}_t$-measurable for all $(t,x) \in [0,\infty) \times \mathbb{R}$,
(3) if the T-wealth X_T is attainable from the t-wealth X_t for some $0 \le t \le T$, then $\mathbb{E}[U(T,X_T)^+|\mathcal{F}_t] < \infty$ almost surely.

Motivated by the papers of Musiela and Zariphopoulou [10] and Henderson and Hobson [4], we will say that U is a forward utility iff

(1) if the T-wealth X_T is attainable from the t-wealth X_t for some $0 \le t \le T$ then

$$U(t,X_t) \ge \mathbb{E}[U(T,X_T)|\mathcal{F}_t] \text{ almost surely,}$$

(2) for each $0 \le t \le T$ and t-wealth X_t there exists an attainable T-wealth X_T^* such that

$$U(t,X_t) = \mathbb{E}[U(T,X_T^*)|\mathcal{F}_t] \text{ almost surely.}$$

Remark 1. At this stage, the reader may be worried about measurability issues. However, since $U(t,\cdot,\omega)$ is concave and finite valued on $(0,\infty)$, it is continuous. Now, if X is a positive $\mathcal{F}_t$-measurable random variable, then there exists a sequence of positive simple $\mathcal{F}_t$-measurable random variables $(X_n)_n$ such that $X_n(\omega) \to X(\omega)$ for each $\omega \in \Omega$ and hence $U(t,X_n(\omega),\omega) \to U(t,X(\omega),\omega)$. But if X_n is of the form $\sum_j x_j \mathbb{1}_{A_j}$ where $x_j > 0$ and $A_j \in \mathcal{F}_t$, then $\omega \to U(t,X_n(\omega),\omega) = \sum_j U(t,x_j,\omega)\mathbb{1}_{A_j}(\omega)$ is $\mathcal{F}_t$-measurable. Therefore, $\omega \to U(t,X(\omega),\omega)$ is also measurable.

Furthemore let us remark on the notion of conditional expectations appearing in the definition of a random utility and forward utility. Recall that the conditional expectation of a non-negative random variable X given a sub-sigma-field $\mathcal{G}$ can always be defined as $\mathbb{E}(X|\mathcal{G}) = \sup_n \mathbb{E}(X \wedge n|\mathcal{G})$. With this notion of conditional expectation, we have a conditional monotone convergence theorem: If $0 \le X_n \uparrow X$ then $\mathbb{E}(X_n|\mathcal{G}) \uparrow \mathbb{E}(X|\mathcal{G})$.

For general random variables, the conditional expectation is defined by linearity. In particular, we let

$$\mathbb{E}[U(T,X_T)|\mathcal{F}_t] = \mathbb{E}[U(T,X_T)^+|\mathcal{F}_t] - \mathbb{E}[U(T,X_T)^-|\mathcal{F}_t]$$

as usual. Note that condition (3) leaves open the possibility that the conditional expectation $\mathbb{E}[U(T,X_T)|\mathcal{F}_t]$ takes the value $-\infty$ with positive probability.

To demonstrate that the concept of a forward utility is not vacuous, we now exhibit a large class of examples:

Example 1. Let $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be increasing, concave, and finite-valued on $(0,\infty)$. Fix a supermartingale $Z \in \mathcal{Z}$, and a trading strategy $\bar{H}$ admissible for the 0-wealth $\bar{X}_0 = 1$, and suppose that

$$\mathbb{E}(Z_T \bar{X}_T|\mathcal{F}_t) = Z_t \bar{X}_t$$

for all $0 \le t \le T$, where

$$\bar{X}_t = 1 + \int_0^t \bar{H}_u \cdot dS_u.$$

Now let

$$U(t,x,\omega) = Z_t(\omega)\bar{X}_t(\omega)\ u\left(\frac{x}{\bar{X}_t(\omega)}\right).$$

We claim that U is a forward utility. Indeed, fix a t-wealth X_t and a T-wealth attainable from X_t. Then by the conditional Jensen inequality

$$\begin{aligned}
\mathbb{E}[U(T, X_T)|\mathcal{F}_t] &= \mathbb{E}\left[Z_T \bar{X}_T u\left(\frac{X_T}{\bar{X}_T}\right)|\mathcal{F}_t\right] \\
&\leq \mathbb{E}[Z_T \bar{X}_T|\mathcal{F}_t] u\left(\frac{\mathbb{E}[Z_T X_T|\mathcal{F}_t]}{\mathbb{E}[Z_T \bar{X}_T|\mathcal{F}_t]}\right) \\
&\leq Z_t \bar{X}_t \, u\left(\frac{X_t}{\bar{X}_t}\right) \\
&= U(t, X_t).
\end{aligned}$$

Finally, note that $X_T^* = X_t \bar{X}_T / \bar{X}_t$ is attainable from X_t with the trading strategy $H_u^* = X_t \bar{H}_u / \bar{X}_t$, and the claim is proven.

We now give one possible motivation for the definition of forward utility. In the classical Merton problem of optimal investment without consumption, we are given the agent's initial wealth $X_0 > 0$, investment horizon $T > 0$, and a (possibly random) utility function $U(T, \cdot)$, and we are asked to find the attainable X_T^* which maximizes the expected utility of terminal wealth $\mathbb{E}[U(T, X_T)]$. A typical application of the Merton problem is in modelling the decision making of an investor planning for retirement. There are some cases of interest, however, in which the introduction of a terminal date seems rather artificial and may not necessarily reflect the considerations of an agent. This is the case, for instance, if we consider a fund manager whose aim is to make her fund's money grow without having any particular investment horizon. Indeed, for this application, one might like a random utility function U with the following time-consistency property: if the strategy $(H_t^1)_{t\in(0,1]}$ maximizes $\mathbb{E}[U(1, X_1)]$ then there exists a strategy $(H_t^2)_{t\in(0,2]}$, with $H_t^1 = H_t^2$ for $t \in (0, 1]$, which maximizes $\mathbb{E}[U(2, X_2)]$. If U is a forward utility, then this time-consistency property is ensured by Lemma 2.4 below.

In section 2 we give a complete characterization of forward utility functions in terms of their convex conjugate functions. As an application, we arrive at a simple characterization of forward utility functions in complete markets.

As the example above shows, the definition of forward utility does not impose very much structure. Indeed, any increasing concave function u can serve as the initial indirect utility $u(x) = U(0, x, \omega)$ and essentially any trading strategy $\bar{H}$ can serve, up to scaling, as the optimal strategy.

In order to impose more structure, we note that for each real $x > 0$, the process $(U(t, x))_{t\geq 0}$ is a supermartingale. Therefore, in section 3 we then proceed to study the case when the market prices are assumed continuous and the the forward utility function is assumed to be almost surely decreasing in time. In this case things are quite different. In particular, we find that all forward utilities, subject to some mild regularity conditions, are of the form $U(t, x, \omega) = u(A_t(\omega), x)$ where the increasing process $(A_t)_{t\geq 0}$ depends only on the model parameters and the deterministic function $(\tau, x) \mapsto u(\tau, x)$ is the concave dual $u(\tau, x) = \inf_{y>0} v(\tau, x) + xy$ of a function v of the form

$$v(\tau, x) = \int_{(0,\infty)} \frac{1}{1-r}(1 - y^{r-1} e^{r(1-r)\tau/2})\nu(dr)$$

for a finite measure ν. In particular, the initial indirect utility $u_0(x) = U(0, x, \omega)$ must be such that $z \mapsto (u_0')^{-1}(e^z)$ is the Laplace transform of a finite measure. Furthermore, the optimal trading strategy of an investor maximizing $\mathbb{E}[U(T, X_T)]$, where U is a decreasing forward utility, is the same, up to a scalar multiple, as that of an investor maximizing $\mathbb{E}[\log(X_T)]$. These results complement the sufficient conditions found in the recent papers of Musiela and Zariphopoulou [11, 12, 13] and of Henderson and Hobson [5].

2. A general characterization

2.1. **Utility functions and their conjugates.** To get started, we now introduce some notation for utility functions and their convex conjugates which we will need in what follows.

Definition 2.1. *Let $\mathcal{U}$ be the set of functions $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ with the following properties:*

(1) $u(x) > -\infty$ *for all* $x > 0$
(2) u *is twice-continuously differentiable, increasing and strictly concave on* $(0, \infty)$
(3) $\lim_{x\downarrow 0} u'(x) = \infty$ *and* $\lim_{x\uparrow\infty} u'(x) = 0$.

Definition 2.2. *Let $\mathcal{V}$ be the set of functions $v : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ with the following properties:*

(1) $v(y) < \infty$ *for all* $y > 0$
(2) v *is twice-continuously differentiable, decreasing and strictly convex on* $(0, \infty)$
(3) $\lim_{y\downarrow 0} v'(y) = -\infty$ *and* $\lim_{y\uparrow\infty} v'(y) = 0$.

We use the notation $\hat{u}$ to denote convex conjugate function of u defined by

$$\hat{u}(y) = \sup_{x>0} u(x) - xy.$$

It is easy to see that if $u \in \mathcal{U}$ then the convex conjugate function $\hat{u} \in \mathcal{V}$. There should be no confusion when the $\hat{}$ notation is used for functions of more than one argument, as the convex conjugation is taken with respect to the wealth variable.

Similarly, we use the notation

$$\check{v}(x) = \inf_{y>0} v(y) + xy$$

so that if $v \in \mathcal{V}$ then $\check{v} \in \mathcal{U}$. Furthermore, the operators $\hat{}$ and $\check{}$ are inverses in that

$$\check{\hat{u}} = u \text{ and } \hat{\check{v}} = v$$

for functions in $\mathcal{U}$ and $\mathcal{V}$ respectively.

Note that for all $x, y > 0$ we have Fenchel's inequality:

$$u(x) \leq \hat{u}(y) + xy.$$

The case of equality in the above inequality has three equivalent formulations:

(1) $y_0 = u'(x_0)$,
(2) $x_0 = -\hat{u}'(y_0)$, and
(3) $u(x_0) = \hat{u}(y_0) + x_0 y_0$.

2.2. **The main results.** We now present the main characterization theorem for forward utility functions. To simplify the formulae, we use the notation U' for $\frac{\partial}{\partial x}U$ and $\hat{U}'$ for $\frac{\partial}{\partial y}\hat{U}$.

Theorem 2.3. *Let U be a random utility function such that $U(t,\cdot,\omega) \in \mathcal{U}$ for all $(t,\omega) \in [0,\infty) \times \Omega$. If U has the property that for all $t \geq 0$ and $\mathcal{F}_t$-measurable positive Y_t, there exists a $Z \in \mathcal{Z}$ such that for all $T \geq t$ the following hold true:*

(1) $\mathbb{E}\left[\hat{U}\left(T, Y_t \frac{Z_T}{Z_t}\right) | \mathcal{F}_t\right] = \hat{U}(t, Y_t)$,

(2) $\mathbb{E}\left[\frac{Z_T}{Z_t}\hat{U}'\left(T, Y_t \frac{Z_T}{Z_t}\right) | \mathcal{F}_t\right] = \hat{U}'(t, Y_t)$, *and*

(3) *the T-wealth $-\hat{U}'\left(T, Y_t \frac{Z_T}{Z_t}\right)$ is attainable from the t-wealth $-\hat{U}'(t, Y_t)$.*

then U is a forward utility.

Suppose U is a forward utility with the property that for all $0 \leq t \leq T$ and t-wealth X_t, the corresponding X_T^* is such that

$$\mathbb{E}[U(T, X_T^*/2)^- | \mathcal{F}_t] < \infty \text{ almost surely.}$$

Then for all $\mathcal{F}_t$-measurable positive Y_t, there exists a $Z \in \mathcal{Z}$ such that properties (1), (2), and (3) hold true.

We begin by proving a lemma, which shows that if U is a forward utility, then the optimal wealth process is independent of the horizon T in a sense to be made precise.

Lemma 2.4. *Let U be a forward utility. Then, for any fixed $t \geq 0$ and t-wealth X_t, there exists an admissible trading strategy $(H_u^*)_{u>t}$ such that*

$$U(u, X_u^*) = \mathbb{E}\left[U(T, X_T^*) | \mathcal{F}_u\right]$$

for all $t \leq u \leq T$, where $X_u^* = X_t + \int_t^u H_s^* \cdot dS_s$.

Proof Lemma 2.4. Fix $0 \leq t \leq u \leq T$ and t-wealth X_t. By assumption, there exists admissible strategy H^*, possibly depending on T, such that

$$U(t, X_t) = \mathbb{E}\left[U(T, X_T^*) | \mathcal{F}_t\right].$$

Notice by the definition of forward utility

$$(*) \qquad \mathbb{E}\left[U(T, X_T^*) | \mathcal{F}_u\right] \leq U(u, X_u^*)$$

as X_T^* is attainable from X_u^*.

Similarly, since since X_u^* is attainable from X_t, we have by the tower property of conditional expectations

$$\begin{aligned}
\mathbb{E}\left[U(u, X_u^*) | \mathcal{F}_t\right] &\leq U(t, X_t) \\
&= \mathbb{E}\left[U(T, X_T^*) | \mathcal{F}_t\right] \\
&= \mathbb{E}\left\{\mathbb{E}\left[U(T, X_T^*) | \mathcal{F}_u\right] \mathcal{F}_t\right\} \\
&\leq \mathbb{E}\left[U(u, X_u^*) | \mathcal{F}_t\right].
\end{aligned}$$

Hence, inequality (*) must hold with almost sure equality, proving the claim. ∎

Proof of Theorem 2.3. Fix $t \geq 0$ and a t-wealth X_t. Let $Y_t = U'(t, X_t)$ and let $Z \in \mathcal{Z}$ be the associated supermartingale. If X_T is attainable from X_t then

$$\begin{aligned}\mathbb{E}[U(T, X_T)|\mathcal{F}_t] &\leq \mathbb{E}\left[\hat{U}\left(T, Y_t \frac{Z_T}{Z_t}\right) + Y_t \frac{Z_T}{Z_t} X_T \Big| \mathcal{F}_t\right] \\ &\leq \hat{U}(t, Y_t) + Y_t X_t \\ &= U(t, X_t).\end{aligned}$$

Finally, we set

$$X_T^* = -\hat{U}'\left(T, Y_t \frac{Z_T}{Z_t}\right),$$

which is attainable by assumption. Substituting X_T^* for X_T above, we see that the first line is actually an equality by Fenchel's inequality, and since $\mathbb{E}[X_T^* Z_T|\mathcal{F}_t] = X_t Z_t$ by assumption, the second line is also an equality. Therefore, U is a forward utility.

Now, conversely, suppose U is a forward utility function. Fix $t \geq 0$ and $\mathcal{F}_t$-measurable positive Y_t, let $X_t = -\hat{U}'(t, Y_t)$, and let $(X_u^*)_{u \geq t}$ be the wealth process starting from $X_t^* = X_t$ such that

$$U(u, X_u^*) = \mathbb{E}[U(T, X_T^*)|\mathcal{F}_u]$$

for all $t \leq u \leq T$. Pick an arbitrary $\bar{Z} \in \mathcal{Z}$ and $Z_s = \bar{Z}_s$ for all $0 \leq s \leq t$ and let

$$Z_u = \frac{U'(u, X_u^*) Z_t}{Y_t}$$

for $u \geq t$. Notice that by construction the random variable $-\hat{U}'\left(T, Y_t \frac{Z_T}{Z_t}\right) = X_T^*$ is attainable from X_t.

Now, we must show that Z as constructed is in $\mathcal{Z}$. Fix $t \leq u < T$ and a u-wealth X_u, and let X_T be attainable from X_u. Then $\epsilon X_T + X_T^*$ is attainable from $\epsilon X_u + X_u^*$ for all $\epsilon > 0$. Hence

$$\mathbb{E}[U(T, \epsilon X_T + X_T^*) - U(T, X_T^*)|\mathcal{F}_u] \leq U(u, \epsilon X_u + X_u^*) - U(u, X_u^*).$$

Since

$$0 < \frac{U(T, \epsilon X_T + X_T^*) - U(T, X_T^*)}{\epsilon} \nearrow U'(T, X_T^*) X_T$$

as $\epsilon \downarrow 0$ because U is increasing and concave, the conditional monotone convergence theorem implies

$$\mathbb{E}[U'(T, X_T^*) X_T|\mathcal{F}_u] \leq U'(u, X_u^*) X_u.$$

Therefore $Z \in \mathcal{Z}$ as claimed, and thus condition (3) is verified.

Now we let $X_u = X_u^*$ in the above argument. Note that $(1+\epsilon) X_T^*$ is attainable from $(1+\epsilon) X_u^*$ for any $\epsilon > -1$. Since

$$2[U(T, X_T^*) - U(T, X_T^*/2)] \geq \frac{U(T, (1+\epsilon) X_T^*) - U(T, X_T^*)}{\epsilon} \searrow U'(T, X_T^*) X_T^*$$

as $\epsilon \uparrow 0$, the conditional monotone convergence theorem implies

$$\mathbb{E}[U'(T, X_T^*) X_T^*|\mathcal{F}_u] \geq U'(u, X_u^*) X_u^*.$$

Therefore, condition (2) is verified.

Finally

$$\begin{aligned}
\mathbb{E}\left[\hat{U}\left(T, Y_t \frac{Z_T}{Z_t}\right)|\mathcal{F}_t\right] &= \mathbb{E}\left[U(T, X_T^*) - Y_t \frac{Z_T}{Z_t} X_T^* \big| \mathcal{F}_u\right] \\
&= U(t, X_t) - Y_t X_t \\
&= \hat{U}(t, Y_t)
\end{aligned}$$

by the condition of equality in Fenchel's inequality. This calculation verifies condition (1) and we are done. □

In a complete market, one can describe all forward utilities as follows:

Corollary 2.5. *Suppose that the market is complete in the sense that $\mathcal{Z}$ contains a single element Z and every T-wealth X_T is attainable from the t-wealth $X_t = \mathbb{E}(Z_T X_T|\mathcal{F}_t)/Z_t$, whenever X_t is finite-valued.*

Suppose U is a random utility with $U(t, \cdot, \omega) \in \mathcal{U}$. Then the following are equivalent:

(1) *U is a forward utility*

(2) *$(V(t, y))_{t \geq 0}$ is a family of generalized[1] martingales such that $V(t, \cdot, \omega) \in \mathcal{V}$ and*

$$U(t, x, \omega) = \check{V}(t, Z_t(\omega)x, \omega).$$

Proof. (1) ⇒ (2) Suppose U is a forward utility, and let $V(t, y, \omega) = \hat{U}(t, y/Z_t(\omega), \omega)$ for each $y > 0$. Condition (1) of Theorem 2.3, together with the fact that Z is unique, imply that $(V(t, y))_{t \geq 0}$ is a family of martingales valued in $\mathcal{V}$.

(2) ⇒ (1) Now given the family $(V(t, y))_{t \geq 0}$ of martingales, note that for all $y > 0$ and $T \geq 0$

$$0 > \frac{V(T, y+\epsilon) - V(T, y)}{\epsilon} \searrow V'(T, y)$$

as $\epsilon \downarrow 0$, by the convexity of $y \mapsto V(T, y)$, so we have

$$\begin{aligned}
\mathbb{E}[V'(T, y)|\mathcal{F}_t] &= \lim_{\epsilon \downarrow 0} \mathbb{E}\left[\frac{V(T, y+\epsilon) - V(T, y)}{\epsilon} |\mathcal{F}_t\right] \\
&= \lim_{\epsilon \downarrow 0} \frac{V(t, y+\epsilon) - V(T, y)}{\epsilon} \\
&= V'(t, y)
\end{aligned}$$

by the conditional monotone convergence theorem.

Let $U(t, x, \omega) = \check{V}(t, Z_t(\omega)x, \omega)$ so that $\hat{U}(t, y) = V(t, y/Z_t)$ and $\hat{U}'(t, y) = V(t, y/Z_t)/Z_t$. Then for any $\mathcal{F}_t$-measurable positive Y_t, we have

$$\begin{aligned}
\mathbb{E}\left[\hat{U}\left(T, Y_t \frac{Z_T}{Z_t}\right)|\mathcal{F}_t\right] &= \mathbb{E}\left[V\left(T, \frac{Y_t}{Z_t}\right)|\mathcal{F}_t\right] \\
&= V\left(t, \frac{Y_t}{Z_t}\right) \\
&= \hat{U}(t, Y_t)
\end{aligned}$$

[1] A generalized martingale M is such that $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$ in the generalized sense for all $0 \leq s \leq t$. In particular, the integrability condition $\mathbb{E}(|M_t - M_s|) < \infty$ is replaced with $\mathbb{E}(|M_t - M_s|\big|\mathcal{F}_s) < \infty$ almost surely.

Similarly,

$$\begin{aligned}\mathbb{E}\left[\frac{Z_T}{Z_t}\hat{U}'\left(T, Y_t\frac{Z_T}{Z_t}\right)|\mathcal{F}_t\right] &= \mathbb{E}\left[\frac{1}{Z_t}V'\left(T, \frac{Y_t}{Z_t}\right)|\mathcal{F}_t\right] \\ &= \frac{1}{Z_t}V'\left(t, \frac{Y_t}{Z_t}\right) \\ &= \hat{U}'(t, Y_t).\end{aligned}$$

Finally, $-\hat{U}'\left(T, Y_t\frac{Z_T}{Z_t}\right)$ is attainable from the t-wealth $-\hat{U}'(t, Y_t)$, since the market is complete, and we are done by Theorem 2.3. □

As a final application of Theorem 2.3, we can consider the dual problem directly:

Corollary 2.6. *If U is a forward utility satisfying with $U(t, \cdot, \omega) \in \mathcal{U}$, then the convex conjugate function $\hat{U}$ satisfies*

$$\hat{U}(t, y) \leq \mathbb{E}\left[\hat{U}\left(y\frac{Z_T}{Z_t}, T\right)|\mathcal{F}_t\right]$$

for any $Z \in \mathcal{Z}$.

Remark 2. Notice that the forward utility property implies

$$U(t, X_t) = \operatorname{ess\,sup} \mathbb{E}[U(T, X_T)|\mathcal{F}_t]$$

where the essential supremum is taken over all X_T attainable from X_t. Since the completion of this paper, Zitkovic [17] has recently taken the above property as the definition of forward utility, by removing the assumption that the optimal wealth X_T^* exists. He has found that U is a forward utility if and only if

$$\hat{U}(t, Y_t) = \operatorname*{ess\,inf}_{Z\in\mathcal{Z}} \mathbb{E}\left[\hat{U}\left(T, Y_t\frac{Z_T}{Z_t}\right)|\mathcal{F}_t\right].$$

3. Decreasing forward utilities

In this section, we explore the consequence of an additional assumption that $t \mapsto U(t, x, \omega)$ is decreasing. It turns out that this assumption greatly restricts the possibilities, and in fact all decreasing forward utility functions can be explicitly characterized.

3.1. **Motivation and assumptions.**

To motivate this study, we note that the strategy of investing no money in the risky assets is admissible for each $x > 0$ and $t \geq 0$, we have

$$\mathbb{E}[U(T, x)|\mathcal{F}_t] \leq U(t, x),$$

and hence that the process $(U(t, x))_{t\geq 0}$ is a super-martingale for all $x > 0$. In particular, if $U(t, x)$ somehow measures the happiness of the investor who has x units of the numéraire asset at time t, then the fact that $(U(t, x))_{t\geq 0}$ is a super-martingale implies that the investor is less happy on average if he does not invest. In what follows, we essentially replace the phrase *on average* with *almost surely* and find that the consequences are substantial.

We now introduce a set of time dependent utility functions that will be needed in what follows. In this section we use a subscript notation to denote partial derivatives, so that $u_x = \frac{\partial u}{\partial x}$, etc.

Definition 3.1. *The set $\mathcal{U}^\circ$ consists of functions $u : [0,\infty) \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that*

(1) $u(t,\cdot) \in \mathcal{U}$, *and*
(2) *the partial derivatives u_t, u_x, u_{xx}, u_{xxx}, and $u_{tx} = u_{xt}$ are defined and continuous on $[0,\infty) \times (0,\infty)$.*

Note that if U is a forward utility function such that $U(\cdot,\cdot,\omega) \in \mathcal{U}^\circ$ then the Doob–Meyer decomposition

$$dU(t,x) = U_t(t,x)dt$$

holds with $U_t(t,x) \le 0$ almost surely.

3.2. **Further assumptions on the market model.** For the rest of this section, we restrict the class of market models to those with continuous price trajectories. We will assume that the prices have the decomposition

$$dS_t^i = S_t^i \left(\mu_t^i \, dt + \sum_{j=1}^d \sigma_t^{i,j} \, dW_t^j \right)$$

for $i \in \{1,\ldots,d\}$ and previsible processes $(\mu_t)_{t\ge 0}$ and $(\sigma_t)_{t\ge 0}$ and k-dimensional Brownian motion $(W_t)_{t\ge 0}$. It is important to note that we do not necessarily suppose that the filtration $(\mathcal{F}_t)_{t\ge 0}$ is generated by the Brownian motion, and hence the market may be incomplete.

For each (t,ω) we let $\Theta_t(\omega)$ be the k-dimensional vector of smallest Euclidean norm such that

$$\sigma_t(\omega)\Theta_t(\omega) = \mu_t(\omega).$$

Note that the vector $\Theta_t(\omega)$ is in the orthogonal complement of the kernel of the matrix $\sigma_t(\omega)$; in other words, $\Theta_t(\omega)$ is in the range of the transpose matrix $\sigma_t(\omega)^{\mathrm{T}}$. Without loss of generality we assume $(\Theta_t)_{t\ge 0}$ is previsible.

We let

$$Z_t^* = \exp\left(-\int_0^t \Theta_u \cdot dW_u - \frac{1}{2}\int_0^t |\Theta_u|^2 du \right)$$

and assume that $\mathbb{E}(Z_t^*) = 1$ for all $t \ge 0$ so that $(Z^*)_{t\ge 0}$ is the density process of the minimal martingale measure. Of course, because of the possibility of incompleteness, the set $\mathcal{Z}$ may contain plenty of other supermartingales.

To prove the main result of this section, we must make the following assumptions on the market.

Assumption 3.2. *The market price of risk process $(\Theta_t)_{t\ge 0}$ is continuous, bounded*[2] *uniformly in $(t,\omega) \in [0,\infty) \times \Omega$, and satisfies the non-degeneracy conditions*

(1) $\Theta_t \ne 0$ *almost surely for all $t \ge 0$, and*
(2) $\int_0^\infty |\Theta_u|^2 \, du = \infty$ *almost surely.*

Assumption 3.2 will be in force throughout the remainder of this section.

In what follows, we will use the notation

$$A_t = \int_0^t |\Theta_s|^2 \, ds$$

[2] The assumption of boundedness is sufficient, but may be too strong in some cases. In fact, the assumption we need for our result is $\mathbb{E}[\mathcal{E}(k\int_0^t \Theta_s \cdot dW_s)] = 1$ for all real $k \ge -1$ and $t \ge 0$.

to denote the mean-variance tradeoff process. We will also use the notation $\mathcal{E}(\cdot)$ for the Doléans–Dade stochastic exponential, so that, for instance,

$$Z_t^* = \mathcal{E}\left(-\int_0^t \Theta_u \cdot dW_u\right).$$

Finally, it will be neater to state results in terms of the amount invested in a particular stock, rather than the number of units. That is, if H is an admissible trading strategy, we can define a new process π by

$$\pi_t^i = S_t^i H_t^i.$$

We will also call π a trading strategy whenever no confusion is possible.

3.3. **The main results.** We are now ready to state and prove the main result of this section.

Theorem 3.3. *Let U be a random utility such that $U(\cdot,\cdot,\omega) \in \mathcal{U}^\circ$. Then the following are equivalent:*

1. *The function U is a forward utility.*
2. *There exists a constant $C \in \mathbb{R}$ and a finite Borel measure ν, supported on the interval $[0,\infty)$ with $\nu\{0\} = 0$ and everywhere finite Laplace transform, such that*

$$\hat{U}(t,y,\omega) = \int_{(0,\infty)} \frac{1}{1-r}\left(1 - y^{1-r} e^{\frac{r(1-r)}{2}A_t(\omega)}\right)\nu(dr) + C \tag{1}$$

almost surely.

We note that the sufficiency of this integral representation was shown by Henderson and Hobson [5] in the case that the log asset price are geometric Brownian motion. Before we prove our main theorem, let us state the following interesting corollaries:

Corollary 3.4. *Let $u : (0,\infty) \to \mathbb{R}$ be a utility function. There exists a forward utility U such that $U(\cdot,\cdot,\omega) \in \mathcal{U}^\circ$ and such that $U(0,x,\omega) = u(x)$ if and only if there exist a finite measure ν such that*

$$(u')^{-1}(y) = \int_{(0,\infty)} y^{-r}\nu(dr),$$

in which case, the function U is unique and is defined by equation (1).

Remark 3. This corollary is in stark contract to Example 1 and indicates how substantial the assumption that the forward utility function is decreasing is.

Corollary 3.5. *If U is a forward utility with $U(\cdot,\cdot,\omega) \in \mathcal{U}^\circ$, then for every $t \geq 0$ and t-wealth X_t, the optimal trading strategy H^* is of the form*

$$\sigma_u^{\mathrm{T}}\pi_u^* = c_u\Theta_u$$

for $u \geq t$, where c_u is a positive scalar-valued random variable.

Remark 4. Corollary 3.5 is a mutual fund theorem with the implication that every investor employing a decreasing forward utility function to decide on optimal asset allocation will choose a positive multiple of the myopic portfolio of a traditional Merton style investor with logarithmic utility. Compare this result with the forward utility constructed in Example 1.

Finally we can refine Corollary 2.6 in the case of decreasing forward utilities.

Corollary 3.6. *If U a decreasing forward utility with $U(\cdot,\cdot,\omega) \in \mathcal{U}^\circ$, then the convex conjugate function $\hat{U}$ satisfies*

$$\hat{U}(t,y) \;=\; \mathbb{E}\left[\hat{U}\left(T, y\frac{Z_T^*}{Z_t^*}\right)|\mathcal{F}_t\right].$$

Remark 5. Corollary 3.6 says that in our particular setup the dual problem admits an optimizer which is the density process of the Föllmer–Schweizer [3] minimal martingale measure. See the paper Kramkov and Schachermayer [7] for the general duality theory for non-random utility functions when the wealth process is constrained to be positive.

Before proving Theorem 3.3, we exhibit some concrete examples of forward utilities. These examples are not surprising, as they are essentially the most common utility functions found in the literature.

Example 2 (Logarithmic Utility). We begin with the simple case where $\nu = \delta_1$ is the Dirac point-mass concentrated at 1. This yields

$$U\big(t,x\big) = \log(x) - \frac{1}{2}A_t$$

Example 3 (CRRA Utility). Now take $R > 0, R \neq 1$ and $\nu = \delta_{R^{-1}}$, the Dirac point-mass at R^{-1}. This choice yields

$$U\big(t,x\big) = \frac{1}{1-R}x^{1-R}\; e^{\frac{(1-R^{-1})}{2}A_t}$$

Example 4 (Time varying relative risk-aversion). Suppose we look for forward utility functions of the form

$$U(t,x) = x^{\gamma_t}C_t$$

for processes $(\gamma_t)_{t\geq 0}$ and $(C_t)_{t\geq 0}$ with differentiable sample paths. Then $(t,\omega) \mapsto \gamma_t(\omega)$ is constant, and C_t is as in Example 3.

Proof of Theorem 3.3. (1)⇒(2)

We first proof that a decreasing forward utility function U satisfies the following non-linear random (random) partial differential equation:

$$U_t(t,x) = \frac{1}{2}|\Theta_t|^2\frac{U_x^2(t,x)}{U_{xx}(t,x)} \text{ a.s. for all } x > 0 \text{ and } t \geq 0.$$

Indeed, fix $x > 0$ and $t \geq 0$. Let π be an admissible trading strategy, and let $X_u = x + \int_t^u \pi_s \cdot (\sigma_s dW_s + \Theta_s ds)$ be the associated discounted wealth process.

By assumption, for each $x > 0$, the semimartingale $(U(t,x))_{t\geq 0}$ has no local-martingale part. Hence, by the generalized Itô formula (see Theorem 3.3.1 of Kunita's book [8], for instance), we have the decomposition

$$\begin{aligned} U(T,X_T) \;&=\; U(t,x) + \int_t^T U_x(u,X_u)\sigma_u^{\mathrm{T}}\pi_u \cdot dW_u \\ &\quad + \int_t^T \left(U_t(u,X_u) + U_x(u,X_u)\sigma_u^{\mathrm{T}}\pi_u\cdot\Theta_u + \frac{1}{2}U_{xx}(u,X_u)|\sigma_u^{\mathrm{T}}\pi_u|^2\right)du. \end{aligned}$$

Since $\big(U(u, X_u)\big)_{u\geq t}$ is a supermartingale, we have

$$\int_t^T \left(U_t(u, X_u) + U_x(u, X_u)\sigma_u^{\mathrm{T}}\pi_u \cdot \Theta_u + \frac{1}{2}U_{xx}(u, X_u)|\sigma_u^{\mathrm{T}}\pi_u|^2 \right) du \leq 0 \text{ a.s.} \tag{2}$$

with equality for the optimal control $\pi = \pi^*$.

Consider the admissible strategy π^+ with corresponding wealth process X_u^+ defined by

$$\pi_u^+ = -\mathbb{1}_{\{X_u^+ \geq x/2\}} \frac{U_x(t,x)}{U_{xx}(t,x)} (\sigma_u^{\mathrm{T}})^{-1}\Theta_u.$$

Using inequality (2), we have

$$\begin{aligned} 0 &\geq \int_t^T \left(U_t(u, X_u^+) + U_x(u, X_u^+)\sigma_u^{\mathrm{T}}\pi_u^+ \cdot \Theta_u + \frac{1}{2}U_{xx}(u, X_u^+)|\sigma_u^{\mathrm{T}}\pi_u^+|^2 \right) du \\ &= \int_t^T \left[U_t(u, X_u^+) - \frac{|\Theta_u|^2}{2}\frac{U_x(t,x)^2}{U_{xx}(t,x)} \mathbb{1}_{\{X_u^+ \geq 0\}} \left(2\frac{U_x(u, X_u^+)}{U_x(t,x)} - \frac{U_{xx}(u, X_u^+)}{U_{xx}(t,x)} \right) \right] du \end{aligned}$$

By the assumed continuity of the partial derivatives of the forward utility, we may let $T \downarrow t$ to conclude that the inequality

$$U_t \leq \frac{1}{2}|\Theta_t|^2 \frac{U_x^2}{U_{xx}} \tag{3}$$

holds almost surely for all $(x,t) \in (0,\infty) \times [0,\infty)$.

Again, fixing $x > 0$ and $t \geq 0$, we appeal to inequality (2), but in the case of equality, so that

$$\begin{aligned} 0 &\geq \int_t^T \left(U_t - \frac{1}{2}|\Theta_u|^2 \frac{U_x^2}{U_{xx}} \right) du \\ &= -\int_t^T \left(\frac{1}{2}U_{xx}|\sigma_u^{\mathrm{T}}\pi_u^*|^2 + U_x\sigma_u^{\mathrm{T}}\pi_u^* \cdot \Theta_u + \frac{1}{2}|\Theta_u|^2 \frac{U_x^2}{U_{xx}} \right) du \\ &= \int_t^T \frac{-1}{2U_{xx}} \left| U_x\Theta_u + U_{xx}\sigma_u^{\mathrm{T}}\pi_u^* \right|^2 du \\ &\geq 0 \end{aligned}$$

where the partial derivatives of U are evaluated at (u, X_u^*) in the integrals, proving the claim.

Now, it is easy to verify by implicit differentiation that the convex conjugate $\hat{U}$ satisfies a linear PDE:

$$\hat{U}_t + \frac{|\Theta_t|^2}{2} y^2 \hat{U}_{yy}.$$

We can solve this PDE by defining a new function $H : [0,\infty) \times \mathbb{R} \times \Omega \to (0,\infty)$ implicitly by

$$H\left(\frac{1}{2}A_t(\omega), \log(y) - \frac{1}{2}A_t(\omega), \omega \right) = -\hat{U}_y(t, y, \omega).$$

Note that by Assumption 3.2 the function H is well-defined for almost all $\omega \in \Omega$. Making the substitution $\tau = \frac{1}{2}A_t$ and $z = \log(y) - \tau$, we have H solves the backward heat equation

$$H_\tau + H_{zz} = 0 \text{ a.s.}$$

Solutions to the backward heat equation are called often space-time harmonic functions. Since the function H is strictly positive, we may use the result of Widder [15, 16] from 1963 characterizing positive space-time harmonic functions:

Theorem 3.7 (Widder)**.** *Let $h : (0, \infty) \times \mathbb{R} \to (0, \infty)$ satisfy*

$$h_\tau + h_{zz} = 0.$$

Then there exists a positive Borel measure ν on $\mathbb{R}$ such that

$$h(t, x) = \int_{\mathbb{R}} e^{-rx - r^2 t} \nu(dr).$$

Hence for almost all $\omega \in \Omega$, there exists a measure $\nu(\cdot, \omega)$ such that

$$\hat{U}_y(y, t, \omega) = -\int_{\mathbb{R}} y^{-r} e^{\frac{r(1-r)}{2} A_t(\omega)} \nu(dr, \omega).$$

But since $\mathcal{F}_0$ was assumed trivial, the random variables $\hat{U}_y(0, y)$ are almost surely constant for each $y > 0$. Since $y \mapsto \hat{U}_y(0, y, \omega)$ is continuous, there exists an almost sure event Ω_0 such that $\omega \mapsto \hat{U}_y(0, \cdot, \omega)$ is constant for each $\omega \in \Omega_0$, and hence we may take ν to be constant since Laplace transforms characterize measures. Finally, by the dual Inada conditions $\lim_{y \downarrow 0} \hat{U}_y(y, t, \omega) = -\infty$ and $\lim_{y \uparrow \infty} \hat{U}_y(y, t, \omega) = 0$ we can conclude that ν is supported on the interval $[0, \infty)$ with $\nu\{0\} = 0$.

(2) $\Rightarrow$ (1) Now to prove the converse. Let ν be a finite measure supported on the interval $[0, \infty)$ with $\nu\{0\} = 0$ and everywhere finite Laplace transform, and let $V : (0, \infty) \times [0, \infty) \times \Omega \to (0, \infty)$ be defined by

$$V(t, y) = \int_{(0, \infty)} \frac{1}{1 - r} \left(1 - y^{1-r} e^{\frac{r(1-r)}{2} A_t}\right) \nu(dr).$$

Note that $y \mapsto V(y, t, \omega)$ is strictly decreasing, strictly convex, and $t \mapsto V(t, y, \omega)$ is decreasing. We need only show that $U = \check{V}$ is a forward utility.

First note that by the assumption that $(\Theta_t)_{t \geq 0}$ is bounded (or see footnote 3.2) and the conditional Fubini theorem, we can check that both $\left(V\big(u, y\frac{Z_u^*}{Z_t^*}\big)\right)_{u \geq t}$ and $\left(\frac{Z_u^*}{Z_t^*} V_y\big(u, y\frac{Z_u^*}{Z_t^*}\big)\right)_{u \geq t}$ are martingales. Finally, we can find an admissible portfolio π such that

$$X_T = \int_{(0, \infty)} y^{-r} \mathcal{E}\left(r \int_t^T \Theta_s \cdot (\Theta_s ds + dW_s)\right) \nu(dr) = -V_y\big(T, y\frac{Z_T^*}{Z_t^*}\big)$$

is attainable from $x = -V_y(t, y)$. Hence the conclusion follows from Theorem 2.3.

□

Remark 6. After a preliminary version of this paper was written, we discovered that Musiela and Zariphopoulou [12] have also employed Widder's theorem to solve the PDE $u_t = \frac{u_x^2}{2u_{xx}}$. These authors in [13] have also employed the generalized Itô formula to show that any solution to a certain stochastic partial differential equation is a forward utility.

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Statistical Laboratory, University of Cambridge, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK

E-mail address: f.berrier@statslab.cam.ac.uk

E-mail address: l.c.g.rogers@statslab.cam.ac.uk

E-mail address: m.tehranchi@statslab.cam.ac.uk