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Explicit solutions of some utility maximization problems in incomplete markets

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Abstract

In this note we prove Hölder-type inequalities for products of certain functionals of correlated Brownian motions. These estimates are applied to the study of optimal portfolio choice in incomplete markets when the investor's utility is of the form $U(X,Y)=g(X)h(Y)$, where X is the investor's wealth and Y is a random factor not perfectly correlated with the market. Explicit solutions are found when g is the exponential, power, or logarithmic utility function.

Keywords: Expected utility; Incomplete markets; Portfolio optimization; Distortion

1. Introduction

In this note we find bounds for expectations of the form $\mathbb{E}\xi\eta$ and $\mathbb{E}\xi\log(\eta)$ where ξ and η are certain functionals of correlated Brownian motions. We then apply these bounds to find explicit solutions to optimal portfolio choice problems in a model of an incomplete financial market.

This analysis is motivated by the following generalization of the classical Merton [10] problem. Let X_t^π be an investor's wealth at time t from employing trading strategy π with initial wealth $X_0=x$. We assume the investor's utility at some fixed future time T is a function $U(X_T^\pi,Y)$ of her wealth and some random factor Y, where Y is (in a sense to be made precise below) not perfectly correlated with the underlying asset prices. The problem is to find the strategies π which maximize the expected utility $\mathbb{E}U(X_T^\pi,Y)$.

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Generalizations of the Merton problem have been extensively studied. For an introduction to the subject, see for instance Karatzas and Shreve [8]. There are two principal approaches. The more general approach appeals to martingale and convex duality arguments and can accommodate very weak assumptions about the dynamics of the underlying asset prices. Indeed, define the convex dual function $\tilde{U}$ by $\tilde{U}(z,y)=\sup_x U(x,y)-xz$. It follows then that we can bound the expected utility by

$$\mathbb{E}U(X_T^\pi, Y) \leqslant \inf_{z,\mathbb{Q}} \ \mathbb{E}\tilde{U}\left(z\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}B_T^{-1}, Y\right) + xz.$$

Here B_T^{-1} denotes the discount factor, and the infimum is taken over $z \geqslant 0$ and measures $\mathbb{Q}$ equivalent to the historical measure $\mathbb{P}$, under which the discounted asset prices are local martingales. Note that if the martingale measure $\mathbb{Q}$ is unique (the market is complete), the dual minimization problem is much easier to solve than the original problem. This generalization of the Merton problem has been studied under very weak assumptions on the asset prices; for instance see [1] or [9].

The other common approach to the Merton problem appeals to the dynamic programming principle. Although this approach is only available under a Markovian assumption on the asset prices, the optimal portfolio can often be expressed in terms of the solution to a related Hamiliton–Jacobi–Bellman equation, and hence may be studied by the techniques of partial differential equations. For instance, suppose the random factor $Y = Y_T$ is given to be the time T value of a diffusion $(Y_t)_{t\geqslant 0}$. Then the value function J, given by

$$J(x,y,t) = \sup_{(\pi_s)_{s\in[t,T]}} \mathbb{E}(U(X_T^\pi, Y_T)\,|\,X_t = x, Y_t = y),$$

formally satisfies the following PDE:

$$\frac{\partial J}{\partial t} + \sup_\pi \mathscr{L}^\pi J = 0, \quad J(x,y,T) = U(x,y),$$

where $\mathscr{L}^\pi$ is the generator of the controlled diffusion process $(X_t^\pi, Y_t)_{t\geqslant 0}$. In fact, this approach was originally employed by Merton [10] to solve the problem in the complete market case with constant market parameters. An advantage of this approach is that explicit solutions are available in some cases.

Zariphopoulou [14] applied the dynamic programming approach to study the optimal portfolio problem in the cases where the utility function is $U(x,y) = (1/\gamma)x^\gamma h(y)$ for $\gamma < 1$, and found that the maximum expected utility is of the form $(\mathbb{E}\xi^{1/\delta})^\delta$ for a random variable ξ depending on Y and the market parameters. More recently, several papers [5,6,7,11] have employed similar methods when the utility function is $U(x,y) = -\mathrm{e}^{-\gamma x}h(y)$ for $\gamma > 0$ and have found that the maximum expected utility is again of the form $(\mathbb{E}\xi^{1/\delta})^\delta$. The constant δ, called the distortion power in the literature, depends on the correlation of the underlying assets, and in the case of power utility, on the risk aversion parameter γ. Their proofs depend crucially on the Markovian structure of market model, and the distortion power δ appears as the exponent of a linearizing transformation for the associated Hamilton–Jacobi–Bellman equation.

The contribution of this note is to provide a systematic account of the role of the distortion power in the cases $U(x,y)=(1/\gamma)x^\gamma h(y)$ for $\gamma<1$, $U(x,y)=-\mathrm{e}^{-\gamma x}h(y)$ for $\gamma>0$, and $U(x,y)=\log(x)h(y)$. The novelty of our approach is that the distortion power arises from simple Hölder-type inequalities. In particular, our results do not require the assumption of Markovian price processes. The crucial assumptions are that the prices are driven by a Wiener process W, the random factor Y is a functional of a Wiener process $\tilde{W}$, and that the correlation of W and $\tilde{W}$ is a fixed constant ρ.

In Section 2 we state Hölder-type inequalities for the expectations $\mathbb{E}\xi\eta$ and $\mathbb{E}\xi\log(\eta)$. In Section 3 we apply the inequalities to optimal portfolio choice for the cases of the exponential and power utility functions, generalizing the results in [5,6,11,13,14]. We also solve the optimal portfolio problem for the logarithmic utility function. In Section 4 we prove the main theorems.

2. The main theorems

Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space supporting correlated standard Wiener processes $W=(W_t)_{t\geqslant 0}$ and $\tilde{W}=(\tilde{W}_t)_{t\geqslant 0}$ with fixed correlation ρ such that $0<|\rho|<1$. Let $(\mathscr{F}_t)_{t\geqslant}$ be the completion of the filtration generated by the pair $(W,\tilde{W})$, and let $\tilde{\mathscr{F}}$ be the completion of the σ-field generated by $\tilde{W}$.

For every $(\mathscr{F}_t)_{t\geqslant 0}$-progressively measurable process $\alpha=(\alpha_t)_{t\geqslant 0}$ such that $\int_0^\infty \alpha_s^2\,\mathrm{d}s<+\infty$ almost surely and for every $\varepsilon\in\mathbb{R}$ we use the following notation:

$$\eta_t^{(\alpha,\varepsilon)}=\exp\left(\frac{\varepsilon\rho^2-1}{2}\int_0^t \alpha_s^2\,\mathrm{d}s+\int_0^t \alpha_s\,\mathrm{d}W_s\right)$$

for $t\geqslant 0$ and $\eta^{(\alpha,\varepsilon)}=\lim_{t\to\infty}\eta_t^{(\alpha,\varepsilon)}$.

We use the notation $\mathbb{E}_{\mathscr{G}}$ for the conditional expectation given the sub-σ-field $\mathscr{G}\subset\mathscr{F}$. In Section 3 we use the notation $\mathbb{E}^{\mathbb{Q}}$ to represent expectation with respect to a measure $\mathbb{Q}\sim\mathbb{P}$.

The following theorems are Hölder-type inequalities for certain functionals of W and $\tilde{W}$. We state them as two theorems to ease the exposition.

Theorem 2.1. *Let $\xi>0$ be an $\tilde{\mathscr{F}}$-measurable random variable and let α be progressively measurable and such that $\int_0^\infty \alpha_s^2\,\mathrm{d}s<+\infty$ almost surely.*

(1) *Let $\varepsilon<0$ and $1/\delta+1/\varepsilon=1$. If $\mathbb{E}\xi^{1/\delta}<+\infty$ we have*

$$\mathbb{E}\xi\eta^{(\alpha,\varepsilon)}\leqslant(\mathbb{E}\xi^{1/\delta})^\delta$$

with equality if

$$\alpha_t=\frac{1}{(1-\varepsilon)\rho}\frac{\beta_t}{\mathbb{E}_{\mathscr{F}_t}\xi^{1/\delta}}\quad\text{where } \xi^{1/\delta}=\mathbb{E}\xi^{1/\delta}+\int_0^\infty \beta_s\,\mathrm{d}\tilde{W}_s.$$

(2) *Let $\varepsilon>0$, $\varepsilon\neq 1$, and $1/\delta+1/\varepsilon=1$. If $\mathbb{E}\xi^{1/\delta}<+\infty$ and $\mathbb{E}\sup_{t\geqslant 0}\eta_t^{(\alpha,\varepsilon)}<+\infty$ then we have*

$$\mathbb{E}\xi\eta^{(\alpha,\varepsilon)}\geqslant(\mathbb{E}\xi^{1/\delta})^\delta$$

with equality if

$$\alpha_t = \frac{1}{(1-\varepsilon)\rho}\,\frac{\beta_t}{\mathbb{E}_{\mathscr{F}_t}\ \xi^{1/\delta}} \quad \textit{where } \xi^{1/\delta} = \mathbb{E}\xi^{1/\delta} + \int_0^\infty \beta_s\, \mathrm{d}\tilde{W}_s.$$

Remark 1. The extra integrability condition in part (2) of Theorem 2.1 is almost necessary. If the integrability condition is dropped, there exist numbers $\varepsilon > 0$, random variables ξ, and processes α with $\int_0^\infty \alpha_s^2\, \mathrm{d}s < +\infty$ almost surely such that

$$\mathbb{E}\xi\eta^{(\alpha,\varepsilon)} < (\mathbb{E}\xi^{1/\delta})^\delta.$$

For instance, letting $\alpha_t = \mathbf{1}_{\{t \leqslant \tau\}}$, where

$$\tau = \inf\{t \geqslant 0 \text{ such that } W_t = -1\},$$

we have $\int_0^\infty \alpha_s^2\, \mathrm{d}s = \tau < +\infty$ almost surely. Hence if $\varepsilon = 1/\rho^2$ and $\xi = 1$, we have $\mathbb{E}\xi\eta^{(\alpha,\varepsilon)} = \mathrm{e}^{-1} < 1$.

The pathology of the above example is not a consequence of working on an infinite time horizon. Indeed, fix a finite horizon $T > 0$, and let $\alpha_t = (T-t)^{-1}\mathbf{1}_{\{t \leqslant \tau\}}$ where

$$\tau = \inf\left\{t \geqslant 0 \text{ such that } \int_0^t (T-s)^{-1}\, \mathrm{d}W_s = -1\right\}.$$

Note that since $\int_0^T (T-s)^{-2}\, \mathrm{d}s = +\infty$ we have $\tau < T$ almost surely and hence $\int_0^T \alpha_s^2\, \mathrm{d}s = (T-\tau)^{-1} - T^{-1} < +\infty$. This example was taken essentially from Dudley [2].

The next theorem handles the cases $\varepsilon = 0$ and $\varepsilon = 1$ excluded in Theorem 2.1.

Theorem 2.2. *Let $\xi > 0$ be an $\tilde{\mathscr{F}}$-measurable random variable and let α be progressively measurable and such that $\int_0^\infty \alpha_s^2\, \mathrm{d}s < +\infty$ almost surely.*

(1) *If $\mathbb{E}\xi\log(\xi)^+ < +\infty$ and $\mathbb{E}\int_0^\infty \alpha_s^2\, \mathrm{d}s < +\infty$ then we have*

$$\mathbb{E}\xi\log(\eta^{(\alpha,0)}) \leqslant \rho^2\mathbb{E}\xi\log\left(\frac{\xi}{\mathbb{E}\xi}\right)$$

with equality if

$$\alpha_t = \rho\,\frac{\beta_t}{\mathbb{E}_{\mathscr{F}_t}\xi} \quad \textit{where } \xi = \mathbb{E}\xi + \int_0^\infty \beta_s\, \mathrm{d}\tilde{W}_s$$

(2) *If $\mathbb{E}|\log(\xi)| < +\infty$ and $\operatorname{ess\,inf}_{(t,\omega)\in\mathbb{R}_+\times\Omega}\eta_t^{(\alpha,1)} > 0$ then we have*

$$\mathbb{E}\xi\eta^{(\alpha,1)} \geqslant \exp(\mathbb{E}\log(\xi))$$

with equality if

$$\alpha_t = -\frac{1}{\rho}\,\beta_t \quad \textit{where } \log(\xi) = \mathbb{E}\log(\xi) + \int_0^\infty \beta_t\, \mathrm{d}\tilde{W}_t.$$

We defer the proofs to Section 4.

Remark 2. The above theorems can easily be extended to the multi-dimensional case. Indeed, suppose W is a standard m-dimensional Wiener process and $\tilde{W}$ is a standard

n-dimensional Wiener process such that $\mathbb{E} W_t \otimes \tilde{W}_t = Rt$ for a $m \times n$ matrix R. Let $\rho^2 = \|RR^{\mathrm{T}}\|$ and

$$\eta^{(\alpha,\varepsilon)} = \exp\left(\frac{\varepsilon\rho^2 - 1}{2} \int_0^\infty \|\alpha_s\|^2 \, \mathrm{d}s + \int_0^\infty \langle \alpha_s, \mathrm{d}W_s \rangle \right)$$

for a progressively measurable $\mathbb{R}^m$-valued process α. Then the conclusions of Theorems 2.1 and 2.2 remain true. However, we choose not to pursue this multi-dimensional generalization here since the inequalities are not sharp in general.

3. Applications to optimal portfolio choice

In this section we solve three utility maximization problems. The results presented here extend the results of Zariphopoulou [13, 14] and of Henderson and Hobson [6]. Furthermore, the results here are proved rather differently. In particular, the method employed here does not make use of any Markovian structure, and therefore it is not necessary to solve a partial differential equation.

We present a simple model of a market consisting of one stock and the bank account. This model encompasses the classical Black–Scholes geometric Brownian motion model, as well as many of the models of stochastic volatility proposed and studied by Fouque et al. [4].

3.1. The market model

As usual we fix a probability space to host our model. In order to avoid introducing more notation, we recycle the notation from Section 2. However, care must be taken when the theorems are applied, as the setting is now slightly different than that of Section 2.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space supporting correlated standard Wiener processes $W = (W_t)_{t \geqslant 0}$ and $\tilde{W} = (\tilde{W}_t)_{t \geqslant 0}$ with fixed correlation ρ such that $0 < |\rho| < 1$. Let $(\mathscr{F}_t)_{t \geqslant}$ be the completion of the filtration generated by the pair $(W, \tilde{W})$, and let $\tilde{\mathscr{F}}$ be the completion of the σ-field generated by $\tilde{W}$.

There are two traded assets, a stock and a bank account, with prices at time $t \geqslant 0$ denoted S_t and B_t, respectively. We assume that the price dynamics are given by

$$\mathrm{d}S_t = S_t(\mu_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}W_t), \quad S_0 = S$$

$$\mathrm{d}B_t = B_t r_t \, \mathrm{d}t, \quad B_0 = 1,$$

where the drift $(\mu_t)_{t \geqslant 0}$, volatility $(\sigma_t)_{t \geqslant 0}$, and spot interest rate $(r_t)_{t \geqslant 0}$ are progressively measurable, bounded processes. We assume that the volatility is bounded away from zero.

We also make the following crucial assumption:

Assumption 3.1. The spot interest rate r_t and the Sharpe ratio $\lambda_t = (\mu_t - r_t)/\sigma_t$ are $\tilde{\mathscr{F}}$-measurable for all $t \geqslant 0$.

The above assumption is restrictive, but is clearly satisfied if the interest rate $r_t = r$ and Sharpe ratio $\lambda_t = \lambda$ are constant, as in the classical setting. Furthermore, the assumption is satisfied if the interest rate $r_t = r$ and the drift $\mu_t = \mu$ are constant, and the volatility σ_t is $\tilde{\mathscr{F}}$-measurable, as in many of the stochastic volatility models studied in [4, 12].

3.2. The random factor

Let Y be a positive random variable corresponding to a random factor to the utility. We make the following assumption:

Assumption 3.2. The random variable $Y > 0$ is $\tilde{\mathscr{F}}$-measurable with moments of all orders.

For our results, the exact form of Y is not important. However, we suggest three motivating examples. (1) Suppose the stochastic volatility model is such that σ_t is $\tilde{\mathscr{F}}$-measurable for all $t \geqslant 0$. We may then take $Y = f(1/T \int_0^T \sigma_t^2 \,\mathrm{d}t)$ to be the payout of an option on the realized volatility. (2) Suppose there exists an auxiliary $\tilde{\mathscr{F}}$-measurable process $(Z_t)_{t\geqslant 0}$ representing the price of a correlated untraded asset. The random factor may then be of the form $Y = f(\sup_{t\in[0,T]} Z_t)$ corresponding to the payout a look-back option on Z. (3) Suppose there the $\tilde{\mathscr{F}}$-measurable process $(Z_t)_{t\geqslant 0}$ represents the temperature at a given location. The random factor may then be of the form $Y = f(\sum_{i=1}^N (Z_{T_i} - 65)^+)$ corresponding to the payout of a cooling degree-day option. Clearly many financially interesting examples fit the present framework.

Remark 3. In many papers, a Markovian structure is imposed on the above set up as follows. An auxillary diffusion $(Z_t)_{t\geqslant 0}$ is assumed to evolve according to

$$\mathrm{d}Z_t = b(t, Z_t)\mathrm{d}t + a(t, Z_t)\mathrm{d}\tilde{W}_t.$$

The parameters of the price dynamics are then assumed to be of the form $\mu_t = \mu(t, Z_t)$, $\sigma_t = \sigma(t, Z_t)$, and $r_t = r(t, Z_t)$, and the random endowment is then assumed to be of the form $Y = f(Z_T)$. This extra Markovian structure is unnecessary for the present analysis.

3.3. The utility maximization problem

Let $X_t = \pi_t + \pi_t^0$ denote an investor's wealth at time $t \geqslant 0$, with π_t units of currency in the stock and π_t^0 units in the bank account. We assume that the processes $(\pi_t)_{t\geqslant 0}$ and $(\pi_t^0)_{t\geqslant 0}$ are progressively measurable. By the self-financing condition, the wealth process evolves according to

$$\mathrm{d}X_t = \pi_t \frac{\mathrm{d}S_t}{S_t} + \pi_t^0 \frac{\mathrm{d}B_t}{B_t}$$
$$= \pi_t\sigma_t(\mathrm{d}W_t + \lambda_t \mathrm{d}t) + r_t X_t \,\mathrm{d}t.$$

By a standard calculation, the wealth process is then given by

$$X_t = B_t \left(x + \int_0^t B_s^{-1}\pi_s\sigma_s(\mathrm{d}W_s + \lambda_s\,\mathrm{d}s) \right), \tag{1}$$

where $B_t = \exp(\int_0^t r_s\,\mathrm{d}s)$. Note that by the boundedness assumptions on the market parameters, it is sufficient to assume that $\int_0^t \pi_s^2\,\mathrm{d}s < +\infty$ almost surely for all $t \geqslant 0$ in order to construct the stochastic integral in Eq. (1). We occasionally use the notation $X_t = X_t^\pi$ to emphasize the dependence of the wealth on the strategy $\pi = (\pi)_{t\geqslant 0}$.

Fixing an initial wealth $x > 0$ and a future date $T > 0$, we assume that the investor's utility at time T is given by $U(X_T, Y)$ for some function U. The investor's goal then is to maximize $\mathbb{E}U(X_T^\pi, Y)$ over a set of admissible strategies $\pi \in \mathscr{A}$ and characterize the optimal strategy.

The set $\mathscr{A}$ of admissible strategies will depend on the particular form of the utility function U. A minimal assumption on the set of strategies is that $\mathbb{E}|U(X_T^\pi, Y)| < +\infty$ in order to properly define expected utility. However, because of "doubling strategies" like the pathological examples mentioned in the Remark 1, it is economically more interesting to choose a set of strategies with a bit more integrability.

For the following propositions, we assume the utility can be written in the separable form $U(x, y) = g(x)h(y)$ where g is an increasing, concave function. Since we have made no assumption on Y, other than being $\tilde{\mathscr{F}}$-measurable, there is no loss taking $h(y) = y$. Furthermore, since we have by iterating expectations

$$\mathbb{E}g(X_T)Y = \mathbb{E}(g(X_T)\mathbb{E}_{\mathscr{F}_T}Y)$$

we may assume that the factor Y is $\mathscr{F}_T$-measurable.

Remark 4. It should be noted that another financially interesting assumption is that the utility function is of the form $U(x, y) = g(x + y)$. In this case, the random factor Y can truly be thought of as a random endowment at time T. Of course, if g is the exponential utility function $g(x) = -\mathrm{e}^{-\gamma x}$, then this form of the utility is equivalent to the separable form considered here.

Proposition 3.3 (Exponential utility). *Suppose the utility function is of the form*

$$U(x, y) = -\mathrm{e}^{-\gamma x} y$$

for some $\gamma > 0$, *and assume that the interest rate* $r_t = r$ *is constant. Let the set of admissible strategies be given by*

$$\mathscr{A} = \{(\pi_t)_{t\geqslant 0} \text{ progressively measurable with } \mathbb{E}\,\sup_{t\in[0,T]} \exp(-\gamma' X_t^\pi) < +\infty$$
$$\text{for some } \gamma' > \gamma\}.$$

The maximum expected utility satisfies

$$\sup_{\pi\in\mathscr{A}} \mathbb{E}U(X_T^\pi, Y) = -\exp(-\gamma x \mathrm{e}^{rT})\left(\mathbb{E}^{\mathbb{Q}} Y^{1-\rho^2}\exp\left(-\frac{(1-\rho^2)}{2}\int_0^T \lambda_s^2\,\mathrm{d}s\right)\right)^{1/(1-\rho^2)},$$

where

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\frac{\rho^2}{2}\int_0^T \lambda_s^2\,\mathrm{d}s - \rho\int_0^T \lambda_s\,\mathrm{d}\tilde{W}_s\right).$$

The supremum is attained for the policy $\pi \in \mathscr{A}$ *given by*

$$\pi_t = \frac{\mathrm{e}^{-r(T-s)}}{\gamma\sigma_t}\left(\lambda_t + \frac{\rho}{\gamma(1-\rho^2)}\frac{\beta_t}{\mathbb{E}^{\mathbb{Q}}_{\mathscr{F}_t} Y^{1-\rho^2}\exp(-(1-\rho^2)/2\int_0^T \lambda_s^2\,\mathrm{d}s)}\right),$$

where

$$Y^{1-\rho^2}\exp\left(-\frac{(1-\rho^2)}{2}\int_0^T \lambda_s^2\,\mathrm{d}s\right) = C + \int_0^T \beta_s(\mathrm{d}\tilde{W}_s + \rho\lambda_s\,\mathrm{d}s).$$

Proof. Define new processes V and $\tilde{V}$ by

$$V_t = W_t + \int_0^t \lambda_s\,\mathrm{d}s \quad \text{and} \quad \tilde{V}_t = \tilde{W}_t + \rho\int_0^t \lambda_s\,\mathrm{d}s.$$

Notice that by the Cameron–Martin–Girsanov theorem V and $\tilde{V}$ are Wiener processes under the measure $\bar{\mathbb{P}}$ where

$$\frac{\mathrm{d}\bar{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T \lambda_s^2\,\mathrm{d}s + \int_0^T \lambda_s\,\mathrm{d}W_s\right).$$

Note that the σ-field generated by $\tilde{V}$ is exactly $\tilde{\mathscr{F}}$ because λ_t is $\tilde{\mathscr{F}}$-measurable for all $t \geqslant 0$ by assumption.

We have

$$\mathbb{E}U(X_T, Y) = \mathbb{E}^{\bar{\mathbb{P}}}\left(-\exp\left(-\gamma x\mathrm{e}^{rT} + \int_0^T (\lambda_s - \mathrm{e}^{r(T-s)}\gamma\pi_s\sigma_s)\,\mathrm{d}V_s - \frac{1}{2}\int_0^T \lambda_s^2\,\mathrm{d}s\right) Y\right)$$

$$\leqslant -\exp(-\gamma x\mathrm{e}^{-rT})\left(\mathbb{E}^{\bar{\mathbb{P}}}\exp\left(-\frac{1-\rho^2}{2}\int_0^T \lambda_s^2\,\mathrm{d}s\right) Y^{1-\rho^2}\right)^{1/(1-\rho^2)}$$

by part (2) of Theorem 2.1 with $\varepsilon = 1/\rho^2$, $\xi = Y\exp(-\frac{1}{2}\int_0^T \lambda_s^2\,\mathrm{d}s)$, and $\alpha_t = \lambda_t - \mathrm{e}^{r(T-t)}\gamma\pi_t\sigma_t$. Note that the integrability assumption of Theorem 2.1 is satisfied since by assumption there is a $\gamma' > \gamma$ such that

$$\mathbb{E}^{\bar{\mathbb{P}}} \sup_{t\in[0,T]} \exp\left(\int_0^t \alpha_s \,\mathrm{d}V_s\right)$$

$$= \mathrm{e}^{\gamma x}\mathbb{E}^{\mathbb{P}}\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\bar{\mathbb{P}}} \sup_{t\in[0,T]} \exp\left(\int_0^t \lambda_s \,\mathrm{d}V_s - \gamma X_t\right)\right)$$

$$\leqslant \mathrm{e}^{\gamma x}\left(\mathbb{E}^{\mathbb{P}}\left(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\bar{\mathbb{P}}} \sup_{t\in[0,T]} \mathrm{e}^{\int_0^t \lambda_s \,\mathrm{d}V_s}\right)^{\gamma'/(\gamma'-\gamma)}\right)^{(\gamma'-\gamma)/\gamma'} \left(\mathbb{E}^{\mathbb{P}} \sup_{t\in[0,T]} \mathrm{e}^{-\gamma' X_t}\right)^{\gamma/\gamma'}$$

is finite. Here we have used the fact that the market price of risk λ_t is bounded and hence the stochastic integrals have exponential moments.

Finally, write $V = \rho\tilde{V} + (1-\rho^2)^{1/2}V^{\perp}$ where $V^{\perp}$ is a standard Wiener process independent of $\tilde{V}$. The measure $\mathbb{Q}$ is then given by

$$\frac{\mathrm{d}\bar{\mathbb{P}}}{\mathrm{d}\mathbb{Q}} = \exp\left(-\frac{1-\rho^2}{2}\int_0^T \lambda_s^2 \,\mathrm{d}s + (1-\rho^2)^{1/2}\int_0^T \lambda_s \,\mathrm{d}V_s^{\perp}\right)$$

and note that $\mathbb{E}^{\mathbb{Q}}_{\tilde{\mathscr{F}}} \mathrm{d}\bar{\mathbb{P}}/\mathrm{d}\mathbb{Q} = 1$ implying the result by iterating expectations. □

Remark 5. The measure $\bar{\mathbb{P}}$ in the above proof is the minimal martingale measure of Föllmer and Schweizer [3]. The measure $\mathbb{Q}$, called the *indifference measure* in [11], is the projection of the minimal martingale measure onto the sigma-field $\tilde{\mathscr{F}}$.

The relation with indifference pricing is as follows. Suppose the utility function is of the form $U(x,y) = g(x-y)$, depending only on the difference of the wealth and the liability. An indifference price for Y is given by the constant P such that

$$\sup_{\pi\in\mathscr{A}} \mathbb{E}(g(X_T^{\pi} - Y) \,|\, X_0 = x) = \sup_{\pi\in\mathscr{A}} \mathbb{E}(g(X_T^{\pi}) \,|\, X_0 = x - P);$$

that is, the agent is indifferent to paying P dollars at time 0 versus facing the random liability Y at time T. In general, the constant C may depend on the initial wealth x. However, if the utility is of the form $g(x) = -\mathrm{e}^{-\gamma x}$ the indifference price is wealth independent. In fact, according the above proposition the indifference price is given by

$$P = \frac{\mathrm{e}^{-rT}}{\gamma(1-\rho^2)} \log\left(\frac{\mathbb{E}^{\mathbb{Q}} \exp\left(\gamma(1-\rho^2)Y - \frac{1}{2}(1-\rho^2)\int_0^T \lambda_s^2 \,\mathrm{d}s\right)}{\mathbb{E}^{\mathbb{Q}} \exp(-\frac{1}{2}(1-\rho^2)\int_0^T \lambda_s^2 \,\mathrm{d}s)}\right).$$

Note that this formula agrees with the one found in [11,6] when $\lambda_t = \lambda$ is constant and $Y = f(Z_T)$ is the payout of a European option written on the $\tilde{\mathscr{F}}$-measurable auxilliary diffusion $(Z_t)_{t\geqslant 0}$.

Remark 6. Proposition 3.3 is also related to the utility optimization with stochastic income studied by Henderson in [5]. Suppose the investor has a source of income separate from her gains from trade, and let Z_t be income rate at time $t \geqslant 0$. Assume that Z_t is $\tilde{\mathscr{F}}$-measurable for $t \geqslant 0$ and that the progressively measurable process $(Z_t)_{t\geqslant 0}$ has exponential moments.

The investor's total wealth $X_t^{\mathrm{total}} = \pi_t + \pi_t^0$ evolves according to

$$\mathrm{d}X_t^{\mathrm{total}} = \pi_t \frac{\mathrm{d}S_t}{S_t} + \pi_t^0 \frac{\mathrm{d}B_t}{B_t} + Z_t\,\mathrm{d}t$$

$$= \pi_t\sigma_t(\mathrm{d}W_t + \lambda_t\mathrm{d}t) + (r_tX_t^{\mathrm{total}} + Z_t)\,\mathrm{d}t$$

and hence we have

$$X_t^{\mathrm{total}} = B_t\left(x + \int_0^t B_s^{-1}\pi_s\sigma_s(\mathrm{d}W_s + \lambda_s\,\mathrm{d}s)\right) + B_t\int_0^T B_s^{-1}Z_s\,\mathrm{d}s$$

$$= X_t + B_t\int_0^T B_s^{-1}Z_s\,\mathrm{d}s.$$

Henderson's result corresponds to letting $Y = \exp(-\gamma\int_0^T \mathrm{e}^{(T-s)r}Z_s\,\mathrm{d}s)$ in Proposition 3.3.

Proposition 3.4 (Power utility). *Suppose the utility function is of the form*

$$U(x,y) = \frac{x^\gamma}{\gamma}\,y$$

for some $\gamma < 1,\ \gamma \neq 0$. *For the case* $0 < \gamma < 1$ *let the set of admissible strategies be given by*

$$\mathscr{A} = \{(\pi_t)_{t\geqslant 0} \text{ progressively measurable with } X_t^\pi > 0 \text{ for all } t\in[0,T]\}.$$

For the case $\gamma < 0$ *let the set of admissible strategies be given by*

$$\mathscr{A} = \left\{\begin{matrix} (\pi_t)_{t\geqslant 0} \text{ progressively measurable with } X_t^\pi > 0 \text{ for all } t\in[0,T] \\ \text{and } \mathbb{E}\ \sup_{t\in[0,T]}(X_t^\pi)^{-\gamma'} < +\infty \text{ for some } \gamma' > |\gamma| \end{matrix}\right\}.$$

The maximum expected utility satisfies

$$\sup_{\pi\in\mathscr{A}} \mathbb{E}U(X_T^\pi, Y) = \frac{x^\gamma}{\gamma}\left(\mathbb{E}^{\mathbb{Q}}Y^{1/\delta}\exp\left(\int_0^T \frac{\gamma}{\delta}\left(r_t + \frac{\gamma}{2(1-\gamma)}\lambda_s^2\right)\mathrm{d}s\right)\right)^\delta,$$

where $\delta = (1-\gamma)/(1-\gamma+\rho^2\gamma)$ *and*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\frac{\rho^2\gamma^2}{2(1-\gamma)^2}\int_0^T \lambda_s^2\,\mathrm{d}s + \frac{\rho\gamma}{1-\gamma}\int_0^T \lambda_s\,\mathrm{d}\tilde{W}_s\right)$$

and is attained for the policy $\pi\in\mathscr{A}$ *given by*

$$\pi_t = \frac{X_t\gamma}{(1-\gamma)\sigma_t}\left(\lambda_t + \rho\delta\frac{\beta_t}{\mathbb{E}^{\mathbb{Q}}_{\mathscr{F}_t}\exp(\int_0^T r_s/\delta\,\mathrm{d}s)Y^{1/\delta}}\right),$$

where

$$\exp\left(\int_0^T r_s/\delta\,\mathrm{d}s\right)Y^{1/\delta} = C + \int_0^T \beta_s\left(\mathrm{d}\tilde{W}_t + \frac{\rho\gamma}{1-\gamma}\lambda_s\,\mathrm{d}s\right)$$

Proof. Define new processes V and $\tilde{V}$ by

$$V_t = W_t + \int_0^t \lambda_s \,\mathrm{d}s \quad \text{and} \quad \tilde{V}_t = \tilde{W}_t + \frac{1}{\rho}\int_0^t \lambda_s \,\mathrm{d}s,$$

which are Wiener processes under the measure $\bar{\mathbb{P}}$ where

$$\frac{\mathrm{d}\bar{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left(-\frac{1}{2\rho^2}\int_0^T \lambda_s^2 \,\mathrm{d}s + \frac{1}{\rho}\int_0^T \lambda_s \,\mathrm{d}\tilde{W}_s\right).$$

Again, the σ-field generated by $\tilde{V}$ is exactly $\tilde{\mathscr{F}}$.

By the assumption that the wealth is positive, we may apply Itô's formula to $\log(X_T)$ to yield

$$X_T = xB_T \exp\left(\frac{-1}{2}\int_0^T \left(\frac{\pi_s\sigma_s}{X_s}\right)^2 \mathrm{d}s + \int_0^T \frac{\pi_s\sigma_s}{X_s}\,\mathrm{d}V_s\right).$$

We have

$$\mathbb{E}U(X_T, Y) = \frac{x^\gamma}{\gamma}\,\mathbb{E}^{\bar{\mathbb{P}}}\exp\left(\frac{-1}{2\gamma}\int_0^T \alpha_s^2 \,\mathrm{d}s + \int_0^T \alpha_s \,\mathrm{d}V_s\right),$$

where $\alpha_t = \gamma\pi_t\sigma_t/X_t$ and $\xi = \exp(\int_0^T(\gamma r_s - \lambda_s^2/2\rho^2)\mathrm{d}s + \int_0^T \lambda_s/\rho \,\mathrm{d}\tilde{W}_s)Y$. Letting $\varepsilon = (\gamma - 1)/\rho^2\gamma$ and noting that $\varepsilon < 0$ when $0<\gamma<1$ and $\varepsilon > 1$ when $\gamma < 0$, we have by Theorem 2.1 that

$$\sup_{\pi\in\mathscr{A}} \mathbb{E}U(X_T^{(\pi,x)}, Y) = \frac{x^\gamma}{\gamma}\left(\mathbb{E}^{\bar{\mathbb{P}}} Y^{1/\delta}\exp\left(\int_0^T \left(\frac{\gamma}{\delta}r_s - \frac{\lambda_s^2}{2\rho^2\delta}\right)\mathrm{d}s + \int_0^T \frac{\lambda_s}{\rho\delta}\,\mathrm{d}\tilde{V}_s\right)\right)^\delta.$$

The result follows by noting that the measure $\mathbb{Q}$ satisfies

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\bar{\mathbb{P}}} = \exp\left(-\int_0^T \frac{\lambda_s^2}{2\rho^2\delta^2}\,\mathrm{d}s + \int_0^T \frac{\lambda_s}{\rho\delta}\,\mathrm{d}\tilde{V}_s\right). \qquad \square$$

Remark 7. Note that the intermediate measure $\bar{\mathbb{P}}$ in the above proof is *not* the minimal martingale measure.

Proposition 3.5 (Logarithmic utility). *Suppose the utility function is of the form*

$$U(x,y) = \log(x)y.$$

Let the set of admissible strategies be given by

$$\mathscr{A} = \left\{ \begin{array}{l} (\pi_t)_{t\geqslant 0} \textit{ progressively measurable with } X_t^\pi > 0 \textit{ for all } t\in[0,T] \\ \textit{and } \mathbb{E}\,|\log(X_T)|^\gamma < +\infty \textit{ for some } \gamma > 1 \end{array} \right\}.$$

The maximum expected utility satisfies

$$\sup_{\pi\in\mathscr{A}} \mathbb{E}U(X_T^\pi, Y) = \mathbb{E}Y\left(\log(x) + \int_0^T \left(r_s + \frac{1}{2}\lambda_s^2\right)\mathrm{d}s \right.$$

$$\left. +\rho\int_0^T \lambda_s \,\mathrm{d}\tilde{W}_t + \rho^2\log\left(\frac{Y}{\mathbb{E}Y}\right)\right)$$

and is attained for the policy $\pi \in \mathscr{A}$ *given by*

$$\pi_t = \frac{X_t}{\sigma_t}\left(\lambda_t + \rho\,\frac{\beta_t}{\mathbb{E}_{\mathscr{F}_t} Y}\right),$$

where $Y = \mathbb{E}Y + \int_0^T \beta_s \,\mathrm{d}\tilde{W}_s$.

Proof. As in the power utility case, define Wiener processes V and $\tilde{V}$ by

$$V_t = W_t + \int_0^t \lambda_s \,\mathrm{d}s \quad \text{and} \quad \tilde{V}_t = \tilde{W}_t + \frac{1}{\rho}\int_0^t \lambda_s \,\mathrm{d}s,$$

under the measure $\bar{\mathbb{P}}$ where

$$\frac{\mathrm{d}\bar{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \exp\left(-\frac{1}{2\rho^2}\int_0^T \lambda_s^2 \,\mathrm{d}s - \frac{1}{\rho}\int_0^T \lambda_s \,\mathrm{d}\tilde{W}_s\right).$$

As before, the σ-field generated by $\tilde{V}$ is $\tilde{\mathscr{F}}$.

Appealing to Theorem 2.2, we have

$$\begin{aligned}
&\mathbb{E}U(X_T, Y)\\
&\quad = \mathbb{E}\log(xB_T)Y + \mathbb{E}^{\bar{\mathbb{P}}}\log\left(\frac{X_T}{xB_T}\right)\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\bar{\mathbb{P}}}\,Y\\
&\quad \leqslant \mathbb{E}\log(xB_T)Y + \rho^2\mathbb{E}^{\bar{\mathbb{P}}}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\bar{\mathbb{P}}}\,Y\log\left(\frac{Y\mathrm{d}\mathbb{P}/\mathrm{d}\bar{\mathbb{P}}}{\mathbb{E}^{\bar{\mathbb{P}}}Y\mathrm{d}\mathbb{P}/\mathrm{d}\bar{\mathbb{P}}}\right)\\
&\quad = \mathbb{E}\log(xB_T)Y + \rho^2\mathbb{E}Y\log\left(\frac{Y\exp(\frac{1}{2\rho^2}\int_0^T \lambda_s^2\,\mathrm{d}s + \frac{1}{\rho}\int_0^T \lambda_s\,\mathrm{d}\tilde{W}_s)}{\mathbb{E}Y}\right). \qquad \square
\end{aligned}$$

Remark 8. The investor is myopic if Y is constant almost surely or if $\rho = 0$.

4. Proof of the main theorems

To prove the theorems we need four lemmas.

Lemma 4.1. *For every progressively measurable* $(\alpha_t)_{t\geqslant 0}$ *such that* $\mathbb{E}\int_0^\infty \alpha_s^2\,\mathrm{d}s < +\infty$ *we have*

$$\mathbb{E}_{\tilde{\mathscr{F}}}\int_0^t \alpha_s \,\mathrm{d}W_s = \rho\int_0^t (\mathbb{E}_{\tilde{\mathscr{F}}}\alpha_s)\,\mathrm{d}\tilde{W}_s$$

for $t \geqslant 0$.

Proof. Let the progressively measurable process $(\beta)_{t\geqslant 0}$ be such that $\mathbb{E}\int_0^\infty \beta_s^2\,\mathrm{d}s < +\infty$ and β_t is $\tilde{\mathscr{F}}$-measurable for all $t \geqslant 0$. By Itô's isometry, Fubini's theorem, and iterating expectations we have

$$\mathbb{E}\int_0^t \alpha_s \, \mathrm{d}W_s \int_0^\infty \beta_s \, \mathrm{d}\tilde{W}_s = \mathbb{E}\rho \int_0^t (\mathbb{E}_{\tilde{\mathscr{F}}}\alpha_s)\beta_s \, \mathrm{d}s$$

$$= \mathbb{E}\rho \int_0^t (\mathbb{E}_{\tilde{\mathscr{F}}}\alpha_s) \, \mathrm{d}\tilde{W}_s \int_0^\infty \beta_s \, \mathrm{d}\tilde{W}_s.$$

The lemma follows by noting that every bounded $\tilde{\mathscr{F}}$-measurable random variable Z is of the form $Z = \mathbb{E}Z + \int_0^\infty \beta_s \, \mathrm{d}\tilde{W}_s$ by the martingale representation theorem. $\square$

Lemma 4.2. *Let* $\alpha = (\alpha_t)_{t \geqslant 0}$ *be a progressively measurable process such that* $\int_0^\infty \alpha_s^2 \, \mathrm{d}s < +\infty$ *almost surely.*

(1) *If* $\varepsilon \leqslant 0$ *we have*

$$\mathbb{E}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta^{(\alpha,\varepsilon)})^{1-\varepsilon} \leqslant 1.$$

(2) *Assume* α *is such that*

$$\mathbb{E}\sup_{t \geqslant 0} \eta_t^{(\alpha,\varepsilon)} < +\infty.$$

If $0 < \varepsilon < 1$ *we have*

$$\mathbb{E}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta^{(\alpha,\varepsilon)})^{1-\varepsilon} \geqslant 1.$$

If $\varepsilon > 1$ *we have*

$$\mathbb{E}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta^{(\alpha,\varepsilon)})^{1-\varepsilon} \leqslant 1.$$

Equality is attained for those processes such that α_t *is* $\tilde{\mathscr{F}}$*-measurable for all* $t \geqslant 0$ *and*

$$\mathbb{E}\exp\left(\frac{-(1-\varepsilon)^2\rho^2}{2} \int_0^\infty \alpha_t^2 \, \mathrm{d}t + (1-\varepsilon)\rho \int_0^\infty \alpha_t \, \mathrm{d}\tilde{W}_t \right) = 1.$$

Proof. First, let α be such that $\int_0^\infty \alpha_s^2 \, \mathrm{d}s$ is bounded. Fix $\varepsilon \in \mathbb{R}$, and note that the random variable $\eta_t = \eta_t^{(\alpha,\varepsilon)}$ has finite moments (of both positive and negative orders) for all $t \geqslant 0$. By Itô's rule, we have

$$\eta_t = 1 + \frac{\varepsilon\rho^2}{2} \int_0^t \eta_s \alpha_s^2 \, \mathrm{d}s + \int_0^t \eta_s \alpha_s \, \mathrm{d}W_s.$$

Taking conditional expections and using Fubini's theorem and Lemma 4.1 we have

$$\mathbb{E}_{\tilde{\mathscr{F}}}\eta_t = 1 + \frac{\varepsilon\rho^2}{2} \int_0^t (\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s\alpha_s^2) \, \mathrm{d}s + \rho \int_0^t (\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s\alpha_s) \, \mathrm{d}\tilde{W}_s$$

almost surely for each $t \geqslant 0$. The right-hand side defines a continuous semimartingale, and hence we may apply Itô's rule again to obtain

$$(\mathbb{E}_{\tilde{\mathscr{F}}}\eta_t)^{1-\varepsilon} = 1 + \frac{(1-\varepsilon)\varepsilon\rho^2}{2}\int_0^t (\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s)^{-\varepsilon-1}((\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s\alpha_s^2)(\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s) - (\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s\alpha_s)^2)\,\mathrm{d}s$$

$$+ (1-\varepsilon)\rho\int_0^t (\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s)^{-\varepsilon}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s\alpha_s)\,\mathrm{d}\tilde{W}_s.$$

By the Cauchy–Schwartz inequality, the drift term is negative for the cases $\varepsilon < 0$ and $\varepsilon > 1$, and the drift term is positive for the case $0 < \varepsilon < 1$.

For more general α define the stopping times

$$\tau_N = \inf\{t \geqslant 0 \text{ such that } \int_0^t \alpha_s^2\,\mathrm{d}s \geqslant N\}.$$

(1) For $\varepsilon \leqslant 0$ we have by two applications of Fatou's lemma

$$\mathbb{E}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta)^{1-\varepsilon} \leqslant \mathbb{E}\lim_{N\to\infty}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta_{\tau_N})^{1-\varepsilon}$$

$$\leqslant \lim_{N\to\infty}\mathbb{E}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta_{\tau_N})^{1-\varepsilon} \leqslant 1.$$

(2) For $\varepsilon > 0$ we have $\eta_{\tau_N} \leqslant \sup_{t\geqslant 0}\eta_t$ and hence $\mathbb{E}_{\tilde{\mathscr{F}}}\eta_{\tau_N} \to \mathbb{E}_{\tilde{\mathscr{F}}}\eta$ almost surely by the dominated convergence theorem.

For $0 < \varepsilon < 1$ we have

$$\mathbb{E}\left(\mathbb{E}_{\tilde{\mathscr{F}}}\sup_{t\geqslant 0}\eta_t\right)^{1-\varepsilon} \leqslant \left(\mathbb{E}\sup_{t\geqslant 0}\eta_t\right)^{1-\varepsilon} < +\infty$$

and again by the dominated convergence theorem we have

$$\mathbb{E}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta)^{1-\varepsilon} = \lim_{N\to\infty}\mathbb{E}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta_{\tau_N})^{1-\varepsilon} \geqslant 1.$$

For $\varepsilon > 1$ we have by Fatou's lemma

$$\mathbb{E}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta)^{1-\varepsilon} \leqslant \lim_{N\to\infty}\mathbb{E}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta_{\tau_N})^{1-\varepsilon} \leqslant 1. \qquad \square$$

Proof of Theorem 2.1. The results are implied by Lemma 4.2 and the following estimates:

(1) If $\varepsilon < 0$, we have by Hölder's inequality with $p = 1/\delta$ and $q = 1-\varepsilon$

$$\mathbb{E}\xi\eta^{(\alpha,\varepsilon)} = \mathbb{E}\xi(\mathbb{E}_{\tilde{\mathscr{F}}}\eta^{(\alpha,\varepsilon)})$$

$$\leqslant (\mathbb{E}\xi^{1/\delta})^{\delta}(\mathbb{E}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta^{(\alpha,\varepsilon)})^{1-\varepsilon})^{1/1-\varepsilon}.$$

(2) If $0 < \varepsilon < 1$, we have by Hölder's inequality with $p = 1/\varepsilon$ and $q = 1/(1-\varepsilon)$

$$\mathbb{E}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta^{(\alpha,\varepsilon)})^{1-\varepsilon} = \mathbb{E}\xi^{-(1-\varepsilon)}(\mathbb{E}_{\tilde{\mathscr{F}}}\xi\eta^{(\alpha,\varepsilon)})^{1-\varepsilon}$$

$$\leqslant (\mathbb{E}\xi^{1/\delta})^{\varepsilon}(\mathbb{E}\xi\eta^{(\alpha,\varepsilon)})^{1-\varepsilon}.$$

If $\varepsilon > 1$, we have by Hölder's inequality with $p = \delta$ and $q = \varepsilon$

$$\begin{aligned}\mathbb{E}\xi^{1/\delta} &= \mathbb{E}(\xi^{1/\delta}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta^{(\alpha,\varepsilon)})^{1/\delta}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta^{(\alpha,\varepsilon)})^{-1/\delta})\\ &\leqslant (\mathbb{E}\xi\eta^{(\alpha,\varepsilon)})^{1/\delta}(\mathbb{E}(\mathbb{E}_{\tilde{\mathscr{F}}}\eta^{(\alpha,\varepsilon)})^{1-\varepsilon})^{1/\varepsilon}. \qquad \square\end{aligned}$$

To prove Theorem 2.2 we first prove the following lemmas.

Lemma 4.3. *Let $\alpha = (\alpha_t)_{t\geqslant 0}$ be a progressively measurable process.*

(1) *If $\mathbb{E}\int_0^\infty \alpha_s^2\,\mathrm{d}s < +\infty$ then we have*

$$\mathbb{E}\exp\left(\frac{1}{\rho^2}\,\mathbb{E}_{\tilde{\mathscr{F}}}\log(\eta^{(\alpha,0)})\right) \leqslant 1.$$

(2) *If $\operatorname{ess\,inf}_{(t,\omega)\in\mathbb{R}_+\times\Omega}\eta_t^{(\alpha,1)} > 0$ then we have*

$$\mathbb{E}\log(\mathbb{E}_{\tilde{\mathscr{F}}}\eta^{(\alpha,1)}) \leqslant 0.$$

Proof. The proof follows the same pattern as the proof of Lemma 4.2.

(1) For $\varepsilon = 0$, first assume $\int_0^\infty \alpha_s^2\,\mathrm{d}s$ is bounded. We have by Lemma 4.1 and Itô's formula we have

$$\begin{aligned}\exp\left(\frac{1}{\rho^2}\,\mathbb{E}_{\tilde{\mathscr{F}}}\log(\eta_t)\right) &= 1+\frac{1}{2\rho^2}\int_0^t \exp\left(\frac{1}{\rho^2}\,\mathbb{E}_{\tilde{\mathscr{F}}}\log(\eta_s)\right)((\mathbb{E}_{\tilde{\mathscr{F}}}\alpha_s)^2-\mathbb{E}_{\tilde{\mathscr{F}}}\alpha_s^2)\,\mathrm{d}s\\ &\quad+\frac{1}{\rho}\int_0^t \exp\left(\frac{1}{\rho^2}\,\mathbb{E}_{\tilde{\mathscr{F}}}\log(\eta_s)\right)(\mathbb{E}_{\tilde{\mathscr{F}}}\alpha_s)\,\mathrm{d}\tilde{W}_s.\end{aligned}$$

for all $t \geqslant 0$. Note that the drift is negative.

Now relax the boundedness assumption and instead assume

$$\mathbb{E}\int_0^\infty \alpha_s^2\,\mathrm{d}s < +\infty.$$

By the martingale inequality and the dominated convergence theorem, we have

$$\mathbb{E}_{\tilde{\mathscr{F}}}\int_0^t \alpha_s\,\mathrm{d}W_s \to \mathbb{E}_{\tilde{\mathscr{F}}}\int_0^\infty \alpha_s\,\mathrm{d}W_s$$

almost surely. The result now follows from Fatou's lemma.

(2) For $\varepsilon = 1$, first assume that $\int_0^\infty \alpha_s^2\mathrm{d}s$ is bounded. By Lemma 4.1 and Itô's formula we have

$$\log(\mathbb{E}_{\tilde{\mathscr{F}}}\eta_t) = \frac{\rho^2}{2}\int_0^t\left(\frac{(\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s\alpha_s^2)(\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s)-(\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s\alpha_s)^2}{(\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s)^2}\right)\mathrm{d}s+\rho\int_0^t \frac{\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s\alpha_s}{\mathbb{E}_{\tilde{\mathscr{F}}}\eta_s}\,\mathrm{d}\tilde{W}_s.$$

Note that the drift is negative.

Again relax the boundedness assumption and instead assume

$$\operatorname{ess\,inf}_{(t,\omega)\in\mathbb{R}_+\times\Omega}\eta_t^{(\alpha,1)} > 0.$$

The result follows from two applications of Fatou's lemma. $\square$

Lemma 4.4. *For positive random variables X and Y such that $\mathbb{E}e^X < +\infty$ and $\mathbb{E}Y\log(Y)^+ < +\infty$ we have*

$$\mathbb{E}XY \leqslant \mathbb{E}Y \log\left(\frac{Y\mathbb{E}e^X}{\mathbb{E}Y}\right)$$

with equality if and only if $X = cY$ almost surely for some constant $c > 0$.

Proof. For positive random variable Z, we have by Jensen's inequality

$$\frac{\mathbb{E}\log(Z)Y}{\mathbb{E}Y} \leqslant \log\left(\frac{\mathbb{E}ZY}{\mathbb{E}Y}\right),$$

with equality if and only if Z is constant almost surely. Letting $Z = e^X/Y$ completes the proof. $\square$

Remark 9. The above inequality is the limit as $p \to +\infty$ and $q \to 1$ of the Hölder inequality

$$p(\mathbb{E}e^{X/p}Y - \mathbb{E}Y) \leqslant p((\mathbb{E}e^X)^{1/p}(\mathbb{E}Y^q)^{1/q} - \mathbb{E}Y)$$

for $1/p + 1/q = 1$.

Proof of Theorem 2.2. The results are then implied by Lemma 4.3 and following estimates:

(1) For $\varepsilon = 0$, we have by Lemma 4.4

$$\frac{1}{\rho^2}\,\mathbb{E}\xi\log(\eta^{(\alpha,0)}) \leqslant \mathbb{E}\xi\log\left(\frac{\xi\mathbb{E}\exp(1/\rho^2\mathbb{E}_{\tilde{\mathscr{F}}}\log(\eta^{(\alpha,0)}))}{\mathbb{E}\xi}\right).$$

(2) For $\varepsilon = 1$, we have by Jensen's inequality

$$\mathbb{E}\xi\eta^{(\alpha,1)} \geqslant \exp(\mathbb{E}\log(\xi) + \mathbb{E}\log(\mathbb{E}_{\tilde{\mathscr{F}}}\eta^{(\alpha,1)})). \qquad \square$$

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