Finance Stoch (2006) 10:553–573
DOI 10.1007/s00780-006-0019-z

Optimal portfolio choice in the bond market

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Received: 14 February 2005 / Accepted: 4 July 2006 /
Published online: 9 September 2006

Abstract We consider the Merton problem of optimal portfolio choice when the traded instruments are the set of zero-coupon bonds. Working within a Markovian Heath–Jarrow–Morton model of the interest rate term structure driven by an infinite-dimensional Wiener process, we give sufficient conditions for the existence and uniqueness of an optimal trading strategy. When there is uniqueness, we provide a characterization of the optimal portfolio as a sum of mutual funds. Furthermore, we show that a Gauss–Markov random field model proposed by Kennedy [Math. Financ. **4**, 247–258(1994)] can be treated in this framework, and explicitly calculate the optimal portfolio. We show that the optimal portfolio in this case can be identified with the discontinuities of a certain function of the market parameters.

Keywords Term structure of interest rates · Malliavin calculus · Utility maximization · Infinite-dimensional stochastic processes

JEL Classification E43

Mathematics Subject Classification (2000) 60H07 · 60H15 · 91B28

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1 Introduction

We consider the problem of optimal portfolio choice when the traded instruments are the set of zero-coupon bonds. In particular, we fix a utility function U and a planning horizon $T > 0$, and consider the functional $J(\phi) = \mathbb{E}^{\mathbb{P}}\, U(X_T^{\phi})$ where X_T^{ϕ} is the accumulated wealth at time T generated by the self-financing trading strategy ϕ. Our goal is to characterize the strategy that maximizes J.

This type of utility maximization problem has a long history in financial economics. A seminal paper of Merton (13) from 1969 provides a solution in the case where the investor can trade continuously in a finite set of stocks and the bank account. In the bond market setting, however, the problem of optimal portfolio choice presents new challenges.

Indeed, there are bonds of so many maturities available to trade that many models assume that there exists a continuum of bonds indexed by their maturity date. Let $P(t, T)$ denote the price at time t of a zero-coupon bond which is worth one unit of money at the maturity T, where $T \geq t$. In the Heath–Jarrow–Morton (HJM) modeling framework proposed in (10), the price process $(P(t, T))_{t\in[0,T]}$ is an Itô process for each $T \geq 0$. We will study the utility maximization problem within the HJM framework.

In the original HJM framework, each of the price processes $(P(t, T))_{t\in[0,T]}$ is driven by the same finite-dimensional Wiener process. This modeling assumption has some shortcomings. For example, in the context of such models, there are typically many strategies which hedge the same claim, but most of these hedging strategies are rather unnatural and probably would never be implemented by a bond trader. See (3) for a discussion of this point. Citing such concerns, as well as the need for models with greater flexibility which can be parsimoniously parametrized, the original HJM framework has been generalized by several authors. For instance, Goldstein (8), Kennedy (12), and Santa-Clara and Sornette (16) have proposed various HJM-type random field models. In such models the bond price processes typically satisfy stochastic differential equations driven by an infinite-dimensional Wiener process. HJM type models with discountinuous bond price sample paths have also been proposed, but we do not address this generalization here.

It is important to note that models driven by an infinite-dimensional Wiener process may be not complete in the usual sense: there typically exist contingent claims that cannot be exactly replicated by a self-financing strategy, even if the martingale measure is unique, and even if the notion of strategy is generalized to allow portfolios of bonds with an infinite number of maturities. The cause of this new type of incompleteness is that when the prices are driven by an infinite-dimensional Wiener process, the volatility cannot be bounded away from zero. This lack of completeness in the presence of a unique martingale measure, though in a slightly different context, has led to the introduction by Björk et al. (2) of the notion of approximate completeness. The approximate completeness of bond market models driven by an infinite-dimensional Wiener process has been studied by De Donno and Pratelli (6) and Taflin (18) where the portfolios that investors are allowed to buy correspond to points in the dual of a Banach space of bond price curves.

The problem of optimal portfolio choice in the bond market has been studied recently by Ekeland and Taflin (7). They work in an HJM framework and prove the existence of an optimal portfolio in two cases: when the driving Wiener process is finite-dimensional and when the Wiener process is infinite-dimensional but the market price of risk is deterministic. Furthermore, they give a representation of the optimal portfolio as a sum of two mutual funds. In this article we build upon the work of Ekeland and Taflin by studying the Merton problem in the case when the driving Wiener process is infinite-dimensional and the bond prices are Markovian. Using the Clark–Ocone formula and convex duality, we give sufficient conditions for the existence of an optimal trading strategy. Furthermore, we prove that the optimal portfolio naturally decomposes as a sum of three mutual funds. The first fund is universal in the sense that each investor in the bond market invests a portion of her wealth in this portfolio, independently of the details of her utility function and planning horizon. The second fund consists of the investor's hedge against fluctations in the market price of risk. This second fund does not appear in Ekeland and Taflin's decomposition since the market price of risk is assumed deterministic. Lastly, the third fund comes from the self-financing constraint and hedges against the stochastic discount factor. Because we are maximizing expected utility of wealth, rather than of discounted wealth, this third fund is slightly different than the fund that appears in (7). Under natural conditions on the bond price volatilities and market price of risk, we show that at time $t \in [0, T]$, this third portfolio consists of bonds with maturities in the interval $[t, T]$.

Finally, we examine in detail the optimal portfolio for a class of Gaussian random field models proposed by Kennedy (12) and studied by Goldstein (8) and by Santa-Clara and Sornette (16). We assume that the market price of risk is deterministic so that we may focus our attention on the first fund of the decomposition. We show that the optimal portfolio in this case can be identified with the discontinuities of a certain function of the market parameters.

The outline of the paper is as follows. In Sect. 2 we recall the various notions that arise in the study of bond markets, including the HJM framework and the Musiela notation. In Sect. 3 we specify a general Markovian HJM model of the infinite-dimensional dynamics of the bond prices. In Sect. 4 we introduce the various notions of strategies needed in the context of this model. In Sect. 5 we present our main results: we solve a Merton utility maximization problem and analyze the optimal strategy. In Sect. 6, we exhibit a nontrivial example of an HJM model which satisfies the conditions of Theorem 5.3 and explicitly construct the optimal portfolio. In Sect. 7, we state some results from Malliavin calculus, including the Clark–Ocone formula, which are used in the proofs of the main theorems. In Sect. 8 we present the proofs of the main results.

2 The HJM framework

In this section we recall the HJM framework, proposed by Heath et al. (10), for modeling the bond market. We include this section for motivation and context; our precise modeling assumptions are spelled out in Sect. 3. We also use the notation introduced here to describe an example HJM model in Sect. 6.

Using the parametrization popularized by Musiela (14), let $f_t(x)$ denote the forward rate at time t for time to maturity x. The forward rates are related to the price $P(t, T)$ at time t of a zero-coupon bond with maturity date $T = t + x$ by the formula

$$f_t(x) = -\frac{\partial}{\partial x} \log P(t, t+x)$$

whenever the derivative exists. In this framework, the risk-neutral dynamics formally satisfy the stochastic partial differential equation

$$\mathrm{d}f_t(x) = \left(\frac{\partial}{\partial x} f_t(x) + a_t(x) \right) \mathrm{d}t + b_t(x) \mathrm{d}W_t^{\mathbb{P}}$$

where the process $(W_t^{\mathbb{P}})_{t \geq 0}$ is a Wiener process for the historical measure $\mathbb{P}$. The drift is given by the famous HJM no-arbitrage condition

$$a_t(x) = b_t(x)\lambda_t + b_t(x) \int_0^x b_t(s)\mathrm{d}s$$

where λ_t is the market price of risk. The random variables $b_t(x)$, λ_t, and $W_t^{\mathbb{P}}$ are allowed to be vector-valued, in which case products are interpreted as standard Euclidean inner products.

Let $B_t = \exp\left(\int_0^t r_s \, \mathrm{d}s \right)$ denote the value at time t of the bank account with initial deposit one dollar, where the short rate is given by $r_t = f_t(0)$. The discounted bond price $\tilde{P}_t(x) = B_t^{-1} P(t, t+x)$ in Musiela notation formally satisfies the stochastic partial differential equation

$$\mathrm{d}\tilde{P}_t(x) = \left(\frac{\partial}{\partial x} \tilde{P}_t(x) - \tilde{P}_t(x) \int_0^x b_t(s)\mathrm{d}s \, \lambda_t \right) \mathrm{d}t - \tilde{P}_t(x) \int_0^x b_t(s)\mathrm{d}s \, \mathrm{d}W_t^{\mathbb{P}}.$$

Letting $\sigma_t(x) = -\tilde{P}_t(x) \int_0^x b_t(s)\mathrm{d}s$ and rewriting the above SPDE in integral form, we see that the discounted bond prices satisfy

$$\tilde{P}_t(x) = \tilde{P}_0(t+x) + \int_0^t \sigma_s(x+t-s)\lambda_s \mathrm{d}s + \int_0^t \sigma_s(x+t-s)\mathrm{d}W_s^{\mathbb{P}} \qquad (2.1)$$

with initial data $\tilde{P}_0(\cdot) = P(0, \cdot)$. It is in this form that we specify the HJM model in the next section.

3 The model specification

In this section, we specify a general Markovian HJM model of the discounted bond prices. Following (7), we take the discounted bond price curve $\tilde{P}_t = \tilde{P}_t(\cdot)$ to be the state variable. We will interpret Eq. (2.1) as an evolution equation in a space F of real-valued functions on $\mathbb{R}_+$. We now list the relevant assumptions on F.

Assumption 3.1 1. The real, infinite-dimensional vector space F is equipped with a norm $\|\cdot\|_F$ for which it becomes a separable Banach space. The topological dual space of F is denoted F^*. Furthermore, the norm obeys the parallelogram law so that F is in fact a Hilbert space. We refrain from introducing a separate notation for the induced inner product.

2. The elements of F are continuous, real-valued functions on $\mathbb{R}_+$. In particular, for every $x \in \mathbb{R}_+$, the evaluation functional

$$\delta_x(f) = f(x)$$

is well-defined as a continuous functional on F; that is, δ_x is an element of F^* for all $x \geq 0$.

3. The semigroup $(S_t)_{t\geq 0}$ is strongly continuous in F, where the left shift operator S_t is defined by

$$(S_t f)(x) = f(t + x).$$

Typical examples of spaces which satisfy Assumption 3.1 are Sobolev spaces. See (4) or (7) for a detailed discussion. Also see Sect. 6 for an example.

We assume that there is exactly one martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$; that is, the market is approximately complete. We therefore specify the model directly under this measure. We will reserve the expectation notation $\mathbb{E}$ for expectation under $\mathbb{Q}$, while expectation under $\mathbb{P}$ will be denoted $\mathbb{E}^{\mathbb{P}}$.

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a real, separable, infinite-dimensional Hilbert space G with inner product $\langle \cdot, \cdot \rangle_G$. We let $(W_t)_{t\geq 0}$ be a Wiener process defined cylindrically on G, and let $(\mathcal{F}_t)_{t\geq 0}$ be the augmentation of the filtration it generates. Without loss of generality, one could take $G = \ell^2$ as in (7) as this amounts to choosing an orthonormal basis of G and working with the coordinates in this basis. We prefer to keep G unspecified. We will identify the dual G^* with G without comment.

We now formulate a model of the risk-neutral discounted price dynamics. In what follows, we let

$$F_+ = \{f \in F : f(x) > 0\}$$

be the positive cone of F. Also, the notation $\mathcal{L}_{\text{HS}}(G, F)$ denotes the space of Hilbert–Schmidt operators from G into F with norm

$$\|A\|_{\mathcal{L}_{\text{HS}}(G,F)} = \left(\sum_n \|Ag_n\|_F^2 \right)^{1/2}$$

for any orthonormal basis $(g_n)_n$ of G.

Assumption 3.2 Let $\sigma(\cdot,\cdot) : \mathbb{R}_+ \times F_+ \to \mathcal{L}_{\mathrm{HS}}(G,F)$ be such that $\sigma(\cdot,f)$ is continuous for all $f \in F_+$ and such that $\sigma(t,\cdot)$ is globally Lipschitz, uniformly in $t \geq 0$.

Additionally, we assume that for all $f \in F_+$ and $t \geq 0$ we have

$$\overline{\mathrm{ran}(\sigma(t,f))} = \{g \in F : g(0) = 0\} \tag{3.1}$$

or equivalently

$$\ker(\sigma(t,f)^*) = \mathrm{span}\{\delta_0\} \subset F^*$$

and

$$\|\sigma(t,f)^*\delta_x\|_G \leq Cf(x) \tag{3.2}$$

for some $C > 0$.

Definition 3.3 *We fix an initial discounted bond price curve $\tilde{P}_0 \in F_+$ once and for all and define the* discounted bond price process $(\tilde{P}_t)_{t\geq 0}$ *as the continuous solution to the evolution equation*

$$\tilde{P}_t = S_t\tilde{P}_0 + \int_0^t S_{t-s}\sigma(s,\tilde{P}_s)\mathrm{d}W_s. \tag{3.3}$$

We use the abbreviation $\sigma_t = \sigma(t,\tilde{P}_t)$.

That the discounted bond price process is well-defined follows from the standard existence and uniqueness theorem for mild solutions of the evolution equation (3.3). See for instance Theorem 7.4 of (5) for a proof.

Remark 3.4 The dynamics of the discounted bond price in this model are genuinely infinite-dimensional in the following sense: for every finite set of maturity dates $T_1,\ldots,T_d$, the submarket consisting of the bank account and those bonds with discounted prices $(\tilde{P}_t(T_1-t),\ldots,\tilde{P}_t(T_d-t))$ is incomplete. Note that this property crucially depends on the infinite dimensionality of the state space F, as well as the infinite dimensionality of the driving Wiener process. Indeed, the rank of the martingale operator σ_t is infinite.

Notice that our model is not a finite-factor model, where a bond price model is said to be a finite-factor model if there exist a deterministic function $g : \mathbb{R}^n \to F$ and an n-dimensional diffusion $(Z_t)_{t\geq 0}$ such that $\tilde{P}_t = g(Z_t)$ almost surely for all $t \geq 0$. However, the condition that a model be genuinely infinite-dimensional in the sense described above is much stronger than not being finite-factor. Indeed, there exist infinite-factor bond price models driven by a one-dimensional Wiener process. A discussion of this phenomenon of hypoellepticity in HJM models can be found in (1).

We will make use of the bounds contained in the following proposition, stated without proof:

Proposition 3.5 *For every $t \geq 0$ and $x \geq 0$, the random variable $\tilde{P}_t(x)$ is strictly positive and*

$$\mathbb{E}\tilde{P}_t(x)^p < +\infty$$

for all real p.

The above proposition allows us to *define* the bank account by the formula

$$B_t = \tilde{P}_t(0)^{-1}.$$

3.1 The market price of risk

We have specified the dynamics of the discounted bond prices under the risk neutral measure $\mathbb{Q}$. For the utility maximization problem considered here, we need to know the dynamics of the bond prices under the equivalent historical measure $\mathbb{P}$. However, since we are working on the finite time horizon $[0, T]$, we need only consider the restriction of $\mathbb{P}$ to the sub-sigma-algebra $\mathcal{F}_T$. We will also denote this restriction by $\mathbb{P}$. We have by Girsanov's theorem that the Radon–Nikodym derivative is of the form

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} = \exp\left(-\frac{1}{2}\int_0^T \|\lambda_s\|_G^2 \mathrm{d}s - \int_0^T \lambda_s \mathrm{d}W_s\right)$$

where $(\lambda_t)_{t\in[0,T]}$ is the G-valued market price of risk process. The process $(W_t^{\mathbb{P}})_{t\in[0,T]}$ defined by

$$W_t^{\mathbb{P}} = W_t + \int_0^t \lambda_s \mathrm{d}s$$

is a cylindrical Wiener process on G under the measure $\mathbb{P}$.

We make the following assumption about the market price of risk:

Assumption 3.6 Let $\lambda(\cdot,\cdot) : \mathbb{R}_+ \times F_+ \to G$ be such that $\lambda(\cdot, f)$ is continuous for all $f \in F_+$ and such that $\lambda(t,\cdot)$ is bounded and globally Lipschitz uniformly in $t \geq 0$. We let $\lambda_t = \lambda(t, \tilde{P}_t)$.

We assume that there exist a subset $F_+^0 \subset F_+$ and a measurable function $\Gamma(\cdot,\cdot) : \mathbb{R}_+ \times F_+^0 \to F^*$ such that $\tilde{P}_t \in F_+^0$ for all $t \geq 0$ almost surely and such that

$$\lambda(t,f) = \sigma(t,f)^*\Gamma(t,f) \tag{3.4}$$

for all $t \geq 0$ and $f \in F_+^0$.

The sudden appearance of the subset F_+^0 will be clarified in the example of Sect. 6. We will make use of the following bound: For every real p, we have

$$\mathbb{E}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^p < +\infty$$

which follows from the assumption that $(\lambda_t)_{t\in[0,T]}$ is bounded.

4 Bond portfolios and trading strategies

By way of motivation, consider an investor holding c_0 units of cash (that is, $B_t^{-1}c_0$ units of the bank account) and c_i units of the bond with maturity T_i for $i = 1, \ldots, N$. Her wealth at time t is given by

$$c_0 + \sum_{i=1}^{N} c_i P(t, T_i) = B_t \left(c_0\delta_0 + \sum_{i=1}^{N} c_i \delta_{T_i - t} \right)(\tilde{P}_t) = B_t\, \phi_t(\tilde{P}_t),$$

where we have used the fact that $\delta_0(\tilde{P}_t) = \tilde{P}_t(0) = B_t^{-1}$. That is, the vector of portfolio weights $(c_0, \ldots, c_N)$ corresponds to a functional $\phi_t \in F^*$.

It is interesting to note that the evaluation functionals span a dense subspace of F^*. Indeed, let $\mathcal{S}$ be the closure of span$\{\delta_x, x \geq 0\}$ in the F^* norm and let $\mathcal{S}^\perp = \{f \in F : \mu(f) = 0 \text{ for all } \mu \in \mathcal{S}\}$. If $f \in \mathcal{S}$ then $f(x) = 0$ for all x; that is, $\mathcal{S}^\perp = \{0\}$ and $\mathcal{S} = \mathcal{S}^{\perp\perp} = F^*$ as claimed. We will call elements of F^* portfolios, and processes taking values in F^* strategies. To be precise, we make the following definition:

Definition 4.1 *An* admissible strategy *is a progressively measurable F^*-valued process $(\phi_t)_{t\geq 0}$ such that*

$$\mathbb{E}\int_0^t \|\sigma_s^*\phi_s\|_G^2 \mathrm{d}s < +\infty$$

for all $t \geq 0$.

Remark 4.2 The definition of admissibility given here is well-suited for the Malliavin calculus techniques we will use. However, there are other ways to define admissible strategies. Indeed, the integrability condition may seem unnatural in light of the original economic problem. A popular alternative is to consider strategies such that the process $(\int_0^t \sigma_s^*\phi_s \mathrm{d}W_s)_{t\geq 0}$ is uniformly bounded from below. As we will see, in the important case where the utility function is finite only on a half-line, the solution of the utility maximization problem is the same with either definition of admissibility. However, if the utility function is finite everywhere, the two definitions of admissibility may give rise to different solutions.

We now formulate a definition of the self-financing condition. It is equivalent to that found in (3) and (7).

Definition 4.3 *An admissible strategy $(\phi_t)_{t\geq 0}$ is* self-financing *if there exists a constant $X_0 \in \mathbb{R}$ such that*

$$\phi_t(\tilde{P}_t) - \int_0^t \sigma_s^*\phi_s \mathrm{d}W_s = X_0$$

for almost all $(t, \omega) \in \mathbb{R}_+ \times \Omega$. The set of admissible self-financing strategies is denoted $\mathcal{A}$.

The integrability condition in Definition 4.1 is sufficient for the stochastic integral in Definition 4.3 to be well-defined.

Note that to each strategy $(\phi_t)_{t\geq 0}$ and initial wealth X_0 we can associate a self-financing strategy $(\psi_t)_{t\geq 0}$ by the rule

$$\psi_t = \phi_t + B_t \left(X_0 + \int_0^t \sigma_s^* \phi_s \mathrm{d}W_s - \phi_t(\tilde{P}_t) \right) \delta_0.$$

The term $\psi_t - \phi_t$ corresponds to the amount of money held in or borrowed from the bank account.

Definition 4.4 *We fix an initial wealth $X_0 \geq 0$ and associate with each self-financing strategy $(\phi_t)_{t\geq 0}$ the* wealth process $(X_t^\phi)_{t\geq 0}$ *given by*

$$X_t^\phi = B_t\, \phi_t(\tilde{P}_t) = B_t \left(X_0 + \int_0^t \sigma_s^* \phi_s \mathrm{d}W_s \right).$$

Note that for every self-financing strategy ϕ, the discounted wealth process $(B_t^{-1} X_t^\phi)_{t\geq 0}$ is a martingale for the equivalent martingale measure $\mathbb{Q}$. Hence, there is no arbitrage in this market.

5 The utility maximization problem

We fix a terminal date $T > 0$, an initial wealth X_0, and a utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and let

$$J(\phi) = \mathbb{E}^{\mathbb{P}}\, U(X_T^\phi)$$

be the expected terminal utility of implementing the strategy ϕ. The investor's goal is then to find an admissible strategy $\phi \in \mathcal{A}$ which maximizes the functional J.

Following (7) we list our assumptions on the utility function U and the inverse marginal utility $I(y) = (U')^{-1}(y)$.

Assumption 5.1 The utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is strictly concave, finite and twice continuously differentiable on an open interval $(\underline{x}, \infty)$ for some $\underline{x} \leq 0$, with the value $\underline{x} = -\infty$ allowed. Moreover, we assume $U'(x) \to \infty$ as $x \searrow \underline{x}$. Letting $\underline{y} = \inf_{x>\underline{x}} U'(x) = \lim_{x\to\infty} U'(x)$, we assume that either $\underline{y} = 0$ or $\underline{y} = -\infty$. Define the decreasing function $I : (\underline{y}, \infty) \to (\underline{x}, \infty)$ by $I(y) = (U')^{-1}(y)$.

Moreover, we assume that there exists some $q > 0$ such that the following growth bounds hold:

$$|I(y)| \leq C(|y|^q + |y|^{-q})$$

and

$$|I'(y)| \leq C(|y|^{q+1} + |y|^{-q-1}).$$

Remark 5.2 In (7) the authors list, as condition B, a series of bounds on the utility function from which they derive the above bounds on the inverse marginal utility.

We have introduced the notations $\underline{x}$ and $\underline{y}$ to treat several interesting cases in a systematic way. For instance, for any *increasing* utility function we would have $\underline{y} = 0$. Thus for the CARA utility $U(x) = -\mathrm{e}^{-\gamma x}$ we have $\underline{x} = -\infty$ and $\underline{y} = 0$ whereas for the CRRA utility $U(x) = x^\gamma/\gamma$ we have $\underline{x} = 0$ and $\underline{y} = 0$. However, for the quadratic "utility" function $U(x) = cx - x^2$, which is decreasing on the interval $(c/2, \infty)$, we would have $\underline{x} = -\infty$ and $\underline{y} = -\infty$. The above bounds are written as to handle simultaneously both cases when $\underline{y} = 0$ and when $\underline{y} = -\infty$.

Theorem 5.3 *Under Assumptions* 3.2 *and* 5.1, *there exists a unique admissible strategy $\bar{\phi} \in \mathcal{A}$ which maximizes J.*

Furthermore, the optimal portfolio decomposes into a sum of three mutual funds

$$\bar{\phi} = \Phi^1 + \Phi^2 + \Phi^3$$

with the following properties:

1. *For every $t \in [0, T]$, the normalized random vector $\Phi^1_t / \|\Phi^1_t\|_{F^*} \in F^*$ is a deterministic function of the market parameters σ_t and λ_t, and is independent of the investor's initial wealth X_0, utility function U, and planning horizon T.*
2. *If the function $\lambda(t, \cdot) : F_+ \to G$ is constant for all $t \geq 0$, then $\Phi^2 = 0$.*
3. *If for every $x \geq 0$, the volatility is such that*

$$\sigma(t, f)^* \delta_x = \sigma(t, g)^* \delta_x$$

whenever $f(s) = g(s)$ for all $0 \leq s \leq x$, then

$$\mathrm{supp}\{\Phi^3_t\} \subset [0, T - t]$$

for all $t \in [0, T]$.

Remark 5.4 The decomposition $\bar{\phi} = \Phi^1 + \Phi^2 + \Phi^3$ can be given a financial interpretation. The first fund Φ^1 is universal in the sense that each investor in this market invests a portion of his wealth in Φ^1. We shall see that it is a multiple of the familiar Merton ratio $\sigma^{*-1}\lambda$. The second fund Φ^2 can be interpreted as the investor's hedge against fluctations in the market price of risk. This portfolio is generally non-zero unless λ_t is deterministic. The mutual fund Φ^3 is unique to the bond market setting. It arises because the risk free asset, the bank account, can be replicated by a portfolio of just-maturing bonds. If the volatility satisfies a certain maturity-mixing condition, satisfied by the Gauss–Markov HJM model for instance, then the portfolio Φ^3 consists of bonds with maturities less than the terminal date T.

We defer the proof of Theorem 5.3 to Sect. 8. In the next section we explicitly compute the optimal portfolio for an example HJM model.

6 A Gauss–Markov example

To fix ideas and to demonstrate that there exist nontrivial models which can be expressed in the above framework, we offer an example in this section corresponding to a Gauss–Markov HJM random field model proposed by Kennedy (12) and further studied by Goldstein (8) and by Santa-Clara and Sornette (16).

The model we analyze is determined by three parameters: twice-differentiable functions $m : \mathbb{R}_+ \to \mathbb{R}$ and $n : \mathbb{R}_+ \to \mathbb{R}_+$, and a positive constant $\alpha > 0$.

Informally, we consider the Gauss–Markov HJM model given by

$$\mathrm{d}f_t(x) = \left(\frac{\partial}{\partial x} f_t(x) + a(x)\right)\mathrm{d}t + b(x)\mathrm{d}W_t^{\mathbb{P}}.$$

The function $b : \mathbb{R}_+ \to G$ describes the instantaneous covariance of the forward rates, and is related to the model parameters by

$$\langle b(x), b(y)\rangle_G = n(x)n(y)\mathrm{e}^{-\alpha|x-y|}.$$

Indeed, let $G = L^2(\mathbb{R}_+)$ and for each $x \geq 0$ let $b(x)$ be the element of G given by

$$b(x,s) = \sqrt{2\alpha}\, n(x)\mathbf{1}_{\{s\geq x\}}\mathrm{e}^{-\alpha(s-x)}.$$

The instantaneous drift $a : \mathbb{R}_+ \to \mathbb{R}$ is related to the model parameters by

$$a(x) = m(x) + n(x)\int_0^x n(s)\mathrm{e}^{-\alpha(s-x)}\mathrm{d}s.$$

This forward rate model can be put in the framework of the discounted bond price models discussed above. In what follows, we identify conditions on the model parameters α, m and n so that there exist functions σ, λ and Γ satisfying Assumptions 3.2 and 3.6 for a suitable choice of state space F. Since the model under consideration is time homogeneous, and since the market price of risk is constant, in this section we will abuse notation and let $\sigma(f) = \sigma(t,f)$, $\lambda = \lambda(t,f)$ and $\Gamma(f) = \Gamma(t,f)$. These functions are related to this forward rate model by the equations

$$\sigma(f)^*\delta_x = -f(x)\int_0^x b(s)\mathrm{d}s$$

$$(\sigma(f)\lambda)(x) = -f(x)\int_0^x m(s)\mathrm{d}s$$

$$\sigma(f)^*\Gamma(f) = \lambda$$

for all $t \geq 0$ and $f \in F_+^0$.

We list here the assumptions on the market parameters.

Assumption 6.1 The functions m, m', m'', n' and n'' decay exponentially at infinity. Moreover, the function n is bounded from below.

For the sake of being concrete, we fix $\beta > 0$ sufficiently small and let the state space be

$$F = \left\{ f : \mathbb{R}_+ \to \mathbb{R};\ \lim_{x\to\infty} f(x) = 0 \text{ and } \int_0^\infty f'(x)^2 e^{\beta x} dx < +\infty \right\},$$

where f' denotes the weak derivative of the absolutely continuous function f. The space F is a separable Hilbert space for the norm $\|f\|_F = \left(\int_0^\infty f'(x)^2 e^{\beta x} dx\right)^{1/2}$ and satisfies Assumption 3.1. The dual space F^* is the completion of the space of finite signed measures μ on $\mathbb{R}_+$ for the norm $\|\mu\|_{F^*} = \left(\int_0^\infty \mu[0,x)^2 e^{-\beta x} dx\right)^{1/2}$.

Proposition 6.2 *For every $f \in F$, the linear operator K_f on G defined by*

$$(K_f g)(x) = -\sqrt{2\alpha} \int_0^\infty \left(f(x) \int_0^{s\wedge x} n(t) e^{-\alpha(s-t)} dt \right) g(s) ds$$

is Hilbert–Schmidt from G into F, and the function $\sigma : F \to \mathcal{L}_{HS}(G,F)$ given by

$$\sigma(f) = K_f \tag{6.1}$$

satisfies Assumption 3.2.

Proof That K_f is Hilbert–Schmidt follows from the computation

$$\begin{aligned}
\|K_f\|^2_{\mathcal{L}_{HS}(G,F)} &= 2\alpha \int_0^\infty \int_0^\infty \left[\frac{\partial}{\partial x} \left(f(x) \int_0^{s\wedge x} n(t) e^{-\alpha(s-t)} dt \right) \right]^2 e^{\beta x} ds\, dx \\
&\le 2 \int_0^\infty f'(x)^2 e^{\beta x} \int_0^x \int_0^x n(s) n(t) e^{-\alpha|t-s|} ds\, dt\, dx + 2 \int_0^\infty f(x)^2 e^{\beta x} n(x)^2 dx \\
&\le (4\alpha^{-1} + 2\beta^{-1}) \|f\|_F^2 \int_0^\infty n(x)^2 dx,
\end{aligned}$$

where we have used the Sobolev-type inequality

$$f(x)^2 = \left(\int_x^\infty f'(s) ds \right)^2 \le \left(\int_x^\infty e^{-\beta s} ds \right) \left(\int_x^\infty f'(s)^2 e^{\beta s} ds \right) \le \beta^{-1} e^{-\beta x} \|f\|_F^2$$

and for the norm of the integral operator the estimate

$$\begin{aligned}\int_0^\infty\int_0^\infty n(t)n(u)\mathrm{e}^{-\alpha|u-t|}\mathrm{d}t\,\mathrm{d}u &\le \frac{1}{2}\int_0^\infty\int_0^\infty (n(t)^2+n(u)^2)\mathrm{e}^{-\alpha|u-t|}\mathrm{d}t\,\mathrm{d}u\\ &= 2\int_0^\infty\int_t^\infty n(t)^2\mathrm{e}^{-\alpha(u-t)}\mathrm{d}u\,\mathrm{d}t\\ &= 2\alpha^{-1}\int_0^\infty n(t)^2\mathrm{d}t.\end{aligned}$$

Furthermore, since $\sigma : F \to \mathcal{L}_{\mathrm{HS}}(G, F)$ is linear, the above bounds show that σ is Lipschitz.

We omit the verification of (3.1) and (3.2). □

Proposition 6.3 *Suppose the initial bond price curve* $\tilde{P}_0 \in F_+$ *satisfies* $\inf_{x\ge 0}\mathrm{e}^{3\beta x/4}\tilde{P}_0(x) > 0$. *Define the subset* $F_+^0 \subset F_+$ *by*

$$F_+^0 = \left\{f \in F_+; \inf_{x\ge 0} e^{\beta x} f(x) > 0\right\}.$$

If $(\tilde{P}_t)_{t\ge 0}$ *is the solution to* (3.3) *with* σ *given by* (6.1)*, then* $\tilde{P}_t \in F_+^0$ *for all* $t \in [0, T]$ *almost surely.*

Proof By the above Sobolev inequality we have $\tilde{P}_t(x) \le \beta^{-1/2}\mathrm{e}^{-\beta x/2}\|\tilde{P}_t\|_F$. Since

$$\tilde{P}_t(x) = \tilde{P}_0(t+x)\exp\left(-\frac{1}{2}\int_0^t \left\|\int_0^{t-s+x} b(u)\mathrm{d}u\right\|^2 \mathrm{d}s + \int_0^t\int_0^{t-s+x} b(u)\mathrm{d}u\,\mathrm{d}W_s\right)$$

we have

$$\exp\left(\int_0^t\int_0^{t-s+x} b(u)\mathrm{d}u\,\mathrm{d}W_s\right) \le \beta^{-1/2}\mathrm{e}^{-\beta x/2}\|\tilde{P}_t\|_F\tilde{P}_0(t+x)^{-1}\exp\left(t\int_0^\infty n(s)^2\mathrm{d}s\right).$$

Now, we may take the probability space Ω as the canonical space of continuous functions taking values in a suitable Banach space E, so that the Wiener process is the coordinate map $W_t(\omega) = \omega(t)$, and such that the space $G \subset E$ is the reproducing kernel Hilbert space for the law of W_1. With this choice of Ω, let $\tilde{P}_t^-$ be defined as the F-valued random variable given by $\tilde{P}_t^-(\omega) = \tilde{P}_t(-\omega)$. It follows then that

$$\tilde{P}_t(x) \ge \beta^{1/2}\tilde{P}_0(t+x)^2\mathrm{e}^{\beta x/2}\|\tilde{P}_t^-\|^{-1}\exp\left(-t\int_0^\infty n(s)^2\mathrm{d}s\right)$$

and the conclusion follows. □

Proposition 6.4 *Let* $\lambda \in G$ *be defined via*

$$\lambda(s) = \frac{1}{\sqrt{2\alpha}}\left[\alpha\frac{m(s)}{n(s)} - \left(\frac{m(s)}{n(s)}\right)'\right].$$

Then

$$(\sigma(f)\lambda)(x) = -f(x)\int_0^x m(s)\mathrm{d}s$$

and there exists a function $\Gamma : F_+^0 \to F^*$ *such that* $\sigma(f)^*\Gamma(f) = \lambda$.

If the function

$$R(s) = \frac{\left(\frac{m(s)}{n(s)}\right)'' - \alpha^2\frac{m(s)}{n(s)}}{n(s)}$$

is of locally bounded variation, then $\Gamma(f) \in F^*$ *can be realized as a signed measure* μ *on* $\mathbb{R}_+$ *which solves the equation*

$$\int_s^\infty f(x)\mu(\mathrm{d}x) = \frac{1}{2\alpha}R(s). \tag{6.2}$$

Proof With the formulas in hand, the verification is a tedious but straightforward integration by parts. Assumption 6.1 and Proposition 6.3 guarantee that the norm $\|\Gamma(f)\|_{F^*}$ can be controlled. □

Remark 6.5 Equation (6.2) can be given an financial interpretation: the points of discontinuity of R correspond to the atoms of the measure μ. In particular, all optimal investors will hold the bonds of relative maturities given by the locations of the upward jumps of this function, and will sell short the bonds given by the downward jumps.

7 Some results from Malliavin calculus

There has been much recent academic interest in financial applications of Malliavin calculus. Here we present several results without proof. One may find a more detailed treatment of the following results in Carmona and Tehranchi (3) and Nualart (15) among others.

The Malliavian derivative is a linear map from a space of random variables to a space of processes. We are concerned with the case where the random variables are elements of $L^p(\Omega; H)$ where H is one of the spaces $\mathbb{R}$, F, G, or the Hilbert–Schmidt operators $\mathcal{L}_{\mathrm{HS}}(F, G)$. The Malliavin derivative of a random variable $\xi \in L^p(\Omega; H)$ is a process $D\xi$ in the space $L^p(\Omega; L^2([0, T]; \mathcal{L}_{\mathrm{HS}}(G, H)))$.

The Malliavin derivative operator is unbounded on $L^p(\Omega; G)$, so we take the approach of defining it first on a core and then extending the definition to the closure of this set in the graph norm topology.

Definition 7.1 Smooth random variables $\xi \in L^p(\Omega; H)$ *are of the form*

$$\xi = \kappa\left(\int_0^T h^1(s)\mathrm{d}W_s, \ldots, \int_0^T h^n(s)\mathrm{d}W_s\right), \tag{7.1}$$

where $h^1, \ldots, h^n \in L^2([0,T]; G)$ are deterministic, and where the infinitely differentiable function $\kappa : \mathbb{R}^n \to H$ is, along with all its derivatives, polynomially bounded. The Malliavin derivative of a smooth random variable is defined to be

$$D_t\xi = \sum_{i=1}^{n} \frac{\partial\kappa}{\partial x_i}\left(\int_0^T h^1(s)\mathrm{d}W_s, \ldots, \int_0^T h^n(s)\mathrm{d}W_s\right) \otimes h^i(t).$$

Moreover, if ξ is the $L^p(\Omega, H)$-limit of a sequence $(\xi_n)_{n\geq 1}$ of smooth random variables such that $(D\xi_n)_{n\geq 1}$ converges in $L^p(\Omega; L^2([0,T]; \mathcal{L}_{\mathrm{HS}}(G,H)))$ we define

$$D\xi = \lim_{n\to\infty} D\xi_n.$$

We use the notation $\mathbb{D}^{1,p}(H)$ to represent the subspace of $L^p(\Omega; H)$ where the derivative can be defined by Definition 7.1. This subspace is a Banach space for the graph norm

$$\|\xi\|_{\mathbb{D}^{1,p}(H)} = \left(\mathbb{E}\|\xi\|_H^p + \mathbb{E}\left(\int_0^T \|D_t\xi\|^2_{\mathcal{L}_{\mathrm{HS}}(G,H)}\mathrm{d}t\right)^{p/2}\right)^{1/p}.$$

Now we come to the Clark–Ocone formula, the crucial result that provides an explicit martingale representation for random variables in $\mathbb{D}^{1,2}(\mathbb{R})$ in terms of the Malliavin derivative. A proof of the infinite-dimensional version of this result can be found in (3). Also, an early application of the Clark–Ocone formula to utility maximization problems can be found in the paper of Karatzas and Ocone (11).

Theorem 7.2 (Clark–Ocone formula) *For every $\mathcal{F}_T$-measurable random variable $\xi \in \mathbb{D}^{1,2}(\mathbb{R})$ we have the representation*

$$\xi = \mathbb{E}\,\xi + \int_0^T \mathbb{E}\{D_t\xi|\mathcal{F}_t\}\mathrm{d}W_t.$$

We close this section with two results that allow us to calculate explicit formulas in what follows. The first one is a generalization of the chain rule in the spirit of Proposition 1.2.3 of Nualart (15):

Proposition 7.3 *Let H_1 and H_2 be real separable Hilbert spaces and let $\mathcal{L}(H_1, H_2)$ denote the Banach space of bounded linear operators from H_1 into H_2. Consider a random variable $\xi \in \mathbb{D}^{1,p}(H_1)$ and a globally Lipschitz function $\kappa : H_1 \to H_2$ with Lipschitz constant C. Then the random variable $\kappa(\xi)$ is in $\mathbb{D}^{1,p}(H_2)$ and* *there exists a random variable Z satisfying the bound $\|Z\|_{\mathcal{L}(H_1,H_2)} \leq C$ almost surely and such that*

$$D\kappa(\xi) = ZD\xi.$$

Remark 7.4 Although the function κ may not be differentiable, there still exists a random variable Z which plays the role of a derivative in the sense of the chain rule. Of course if κ is Fréchet differentiable, then $Z = \nabla\kappa(\xi)$ is its Fréchet derivative evaluated at ξ. In Sect. 8 we use this result in the cases $\kappa = \lambda(t,\cdot) : F \to G$ and $\kappa = \sigma(t,\cdot) : F \to \mathcal{L}_{\mathrm{HS}}(G,F)$.

The second result which we state without proof is the infinite dimensional analogue of (1.46) of Nualart (15).

Proposition 7.5 *If the adapted continuous square-integrable process $(\alpha_t)_{t\in[0,T]}$ is such that for all $t \in [0,T]$ the random variable $\alpha_t \in \mathbb{D}^{1,p}(\mathcal{L}_{\mathrm{HS}}(G,H))$ is differentiable, for $p \geq 2$, then*

$$D_t \int_0^T \alpha_s \mathrm{d}W_s = \alpha_t + \int_t^T D_t\alpha_s \mathrm{d}W_s.$$

8 Proofs of the main results

In this section we give the proofs of the main results. Recall that the inverse marginal utility function $I = (U')^{-1}$ plays a crucial role in the study of the optimal investment problem. We begin with a duality lemma.

Let the utility function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and inverse marginal utility $I : (\underline{y},\infty) \to (\underline{x},\infty)$ satisfy Assumption 5.1.

Lemma 8.1 *Fix $X_0 > 0$. There exists a unique real number z_0 such that*

$$\mathbb{E}^{\mathbb{Q}} B_T^{-1} I\left(z_0 B_T^{-1}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) = X_0.$$

Furthermore, for each random variable X with $\mathbb{E}^{\mathbb{Q}}\, B_T^{-1}X = X_0$ we have

$$\mathbb{E}^{\mathbb{P}}\, U(X) \leq \mathbb{E}^{\mathbb{P}}\, U \circ I(\eta)$$

where $\eta = z_0 B_T^{-1}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$.

Proof By Assumption 5.1 there is a constant $C>0$ such that $|I(y)|<C(|y|^q+|y|^{-q})$, where again, the bound is designed to handle both cases for $\underline{y} = 0$ and $\underline{y} = -\infty$. The density $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P}$ and the discounted bond price $\tilde{P}_t(0)$ have moments of all negative orders by Proposition 3.5, and in particular

$$\mathbb{E}^{\mathbb{Q}} B_T^{-1} I\left(z B_T^{-1}\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) < +\infty$$

for all $z \in \mathbb{R}$. Since the function $I : (\underline{y},\infty) \to (\underline{x},\infty)$ is continuous and decreasing, the function $z \mapsto \mathbb{E}^{\mathbb{Q}} B_T^{-1} I(zB_T^{-1}\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P})$ is continuous and decreasing by the monotone convergence theorem, and hence invertible on its range. The number $X_0 > 0$ is in fact contained in the range since $\lim_{y \to \underline{y}} I(y) = +\infty$ and $\lim_{y\to\infty} I(y) = \underline{x} \leq 0$.

Expanding U into a Taylor series about the point $I(\eta)$ and recalling the assumption of concavity we have

$$\begin{aligned}\mathbb{E}^{\mathbb{P}} U(X) &\leq \mathbb{E}^{\mathbb{P}}\ U \circ I(\eta) + \mathbb{E}^{\mathbb{P}}\ \eta(X - I(\eta)) \\ &= \mathbb{E}^{\mathbb{P}}\ U \circ I(\eta) + z_0 \mathbb{E}^{\mathbb{Q}}\ (B_T^{-1} X - B_T^{-1} I(\eta)) \\ &\leq \mathbb{E}^{\mathbb{P}}\ U \circ I(\eta),\end{aligned}$$

completing the proof. □

The main theorem is then proven if we can show that there exists an admissible strategy $\bar{\phi}$ such that $X_T^{\bar{\phi}} = I(\eta)$. See the article of Goldys and Musiela (9) for an analytic approach to this type of question. We approach this question via the Clark–Ocone formula. To this end, we state the following lemma. We omit the proof, but the interested reader can find the proof for the case $p = 2$ in (4), Lemma 5.3.

Lemma 8.2 *For every $p \geq 2$ and all $t \geq 0$ we have $\tilde{P}_t \in \mathbb{D}^{1,p}(F)$.*

We now prove a representation formula for the derivative $D\tilde{P}_t$. We appeal to Skorohod's theory of strong random operators as developed in Skorohod (17). A strong random operator from F into a separable Hilbert space H is an H-valued stochastic process $(Z(f))_{f\in F}$ indexed by F which is linear in $f \in F$. A strong operator process $(Z_t(f))_{t\geq 0, f\in F}$ is similarly defined. If such a process is adapted and if $H = \mathcal{L}_{\mathrm{HS}}(G, F)$, then by setting

$$\left[\int_0^t Z_s \cdot \mathrm{d}W_s\right](f) = \int_0^t Z_s(f) \mathrm{d}W_s$$

we define a strong random operator $\int_0^t Z_s \cdot \mathrm{d}W_s$ from F into F.

The following proposition gives a useful representation formula for the Malliavin derivative of the discounted bond price. See Sect. 5.5.2 of (4) for a proof.

Proposition 8.3 *The Malliavin derivative $D\tilde{P}_t$ is given by*

$$D_s \tilde{P}_t = Y_{s,t} \sigma(s, \tilde{P}_s)$$

for $s \in [0, t]$, where the strong operator process $(Y_{s,t})_{0\leq s\leq t}$ is the solution to the integral equation

$$Y_{s,t} = S_{t-s} + \int_s^t S_{t-u} \nabla \sigma_u Y_{s,u} \cdot \mathrm{d}W_u$$

for $t \geq s$, where $\nabla\sigma_t$ is that $\mathcal{L}(F, \mathcal{L}_{\mathrm{HS}}(G, F))$-valued random variable such that $D\sigma(t, \tilde{P}_t) = \nabla\sigma_t D\tilde{P}$.

Corollary 8.4 *For every $p \geq 2$ we have $B_T^{-1} = \tilde{P}_T(0) \in \mathbb{D}^{1,p}(\mathbb{R})$.*

Lemma 8.5 *For every $p \geq 2$ we have $\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathbb{D}^{1,p}(\mathbb{R})$.*

Proof Recall that the density is given by the exponential

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(\frac{1}{2}\int_0^T \|\lambda(s,\tilde{P}_s)\|_G^2 \mathrm{d}s + \int_0^T \lambda(s,\tilde{P}_s)\mathrm{d}W_s\right).$$

Using the boundedness assumption on the function λ we have

$$D_t\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\left(\lambda(t,\tilde{P}_t) + \int_t^T \nabla\lambda_s D_t\tilde{P}_s \cdot (\mathrm{d}W_s + \lambda(s,\tilde{P}_s)\mathrm{d}s)\right),$$

where $\nabla\lambda_s$ is the bounded $\mathcal{L}(F,G)$-valued random variable such that $D\lambda(s,\tilde{P}_s) = \nabla\lambda_s D\tilde{P}_s$. □

Corollary 8.6 *For every $p \geq 2$ we have $\eta = z_0 B_T^{-1}\mathrm{d}\mathbb{Q}/\mathrm{d}\mathbb{P} \in \mathbb{D}^{1,p}(\mathbb{R})$.*

Lemma 8.7 *For every $p \geq 2$ we have $I(\eta) \in \mathbb{D}^{1,p}(\mathbb{R})$ and $DI(\eta) = I'(\eta)D\eta$.*

Proof Note that there is nothing to prove if $z_0 = 0$ as $I(\eta)$ is constant. So we may assume $z_0 \neq 0$. The chain rule is not directly applicable because of the singularities $\lim_{y\to\underline{y}} I(y) = \infty$ and $\lim_{y\to\underline{y}} I'(y) = -\infty$. We first find a sequence of Malliavin-differentiable random variables $I_n(\eta)$ which converge to $I(\eta)$ in $L^p(\Omega;\mathbb{R})$ such that $\mathbb{E}\left(\int_0^T \|D_s I_n(\eta)\|_G^2 \mathrm{d}s\right)^{p/2}$ is uniformly bounded. By the growth assumption on the utility function and the moment bounds in Proposition 3.5 we have

$$\begin{aligned}\mathbb{E}\,|I(\eta)|^p &\leq C\mathbb{E}\,(|\eta|^{-pq} + |\eta|^{pq})\\ &= C\left(|z_0|^{-pq}\mathbb{E}\,\tilde{P}_T(0)^{-pq}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^{-pq} + |z_0|^{pq}\mathbb{E}\,\tilde{P}_T(0)^{pq}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)^{pq}\right)\\ &< +\infty.\end{aligned}$$

We have to consider two cases for the two limiting values of $y \to \underline{y}$, when $\underline{y} = 0$ and when $\underline{y} = -\infty$. If $\underline{y} = 0$, we let

$$I_n(y) = \begin{cases} I(y) & \text{if } y > \frac{1}{n} \\ I(\frac{1}{n}) & \text{if } y \leq \frac{1}{n} \end{cases}$$

whereas if $\underline{y} = -\infty$, we set

$$I_n(y) = \begin{cases} I(y) & \text{if } y > -n \\ I(-n) & \text{if } y \leq -n \end{cases}$$

and noting that $|I_n(y) - I(y)|^p < 2^p I(y)^p$ since I is decreasing, we have $\mathbb{E}|I_n(\eta) - I(\eta)|^p \to 0$ by the dominated convergence theorem.

Now we show that $\mathbb{E}\left(\int_0^T \|D_s I_n(\eta)\|_G^2 \mathrm{d}s\right)^{p/2}$ is uniformly bounded. Since I_n is Lipschitz we can apply Proposition 7.3 yielding

$$\mathbb{E}\left(\int_0^T \|D_s I_n(\eta)\|_G^2\right)^{p/2} \mathrm{d}s = \mathbb{E}|I_n'(\eta)|^p \left(\int_0^T \|D_s\eta\|_G^2 \mathrm{d}s\right)^{p/2}$$
$$\leq \left(\mathbb{E}|I_n'(\eta)|^{2p}\right)^{1/2} \left(\mathbb{E}\left(\int_0^T \|D_s\eta\|_G^2 \mathrm{d}s\right)^p\right)^{1/2}$$

where

$$I_n'(y) = \begin{cases} I'(y) & \text{if } y > \frac{1}{n} \\ 0 & \text{if } y \leq \frac{1}{n} \end{cases}$$

or

$$I_n'(y) = \begin{cases} I'(y) & \text{if } y > -n \\ 0 & \text{if } y \leq -n \end{cases}$$

depending, of course, on the value of $\underline{y}$. The uniform bound follows from the estimate

$$\mathbb{E}|I_n'(\eta)|^{2p} \leq C^{2p}\mathbb{E}\left(|\eta|^{-2p(q+1)} + |\eta|^{2p(q+1)}\right) < +\infty.$$

In fact, for $\underline{y} = 0$, for every $p \geq 2$ we have $DI_n(\eta) \to I'(\eta)D\eta$ in $L^p(\Omega, L^2([0,T];G))$ since

$$\mathbb{E}\left(\int_0^T \|D_t I_n(\eta) - I'(\eta)D_t\eta\|_G^2 \mathrm{d}t\right)^{p/2} = \mathbb{E}\left(\int_0^T \|(I_n'(\eta) - I'(\eta))D_t\eta\|_G^2 \mathrm{d}t\right)^{p/2}$$
$$= \mathbb{E}\,\mathbf{1}_{\{\eta \leq \frac{1}{n}\}}|I'(\eta)|^p \left(\int_0^T \|D_t\eta\|_G^2 \mathrm{d}t\right)^{p/2}$$

converges to zero by the dominated convergence theorem. The same result clearly holds for $\underline{y} = -\infty$ since

$$\mathbb{E}\left(\int_0^T \|D_t I_n(\eta) - I'(\eta)D_t\eta\|_G^2 \mathrm{d}t\right)^{p/2} = \mathbb{E}\left(\int_0^T \|(I_n'(\eta) - I'(\eta))D_t\eta\|_G^2 \mathrm{d}t\right)^{p/2}$$
$$= \mathbb{E}\,\mathbf{1}_{\{\eta \leq -n\}}|I'(\eta)|^p \left(\int_0^T \|D_t\eta\|_G^2 \mathrm{d}t\right)^{p/2}$$

also converges to zero by dominated convergence. $\square$

Corollary 8.8 *We have $B_T^{-1}I(\eta) \in \mathbb{D}^{1,2}(\mathbb{R})$ and*

$$D_t B_T^{-1} I(\eta) = B_T^{-1}\eta I'(\eta)\lambda_t - B_T^{-1}\eta I'(\eta) \int_t^T \nabla\lambda_s D_t \tilde{P}_s (\mathrm{d}W_s + \lambda_s \mathrm{d}s)$$

$$+(I(\eta) + \eta I'(\eta)) D_t \tilde{P}_T(0).$$

Combining the previous representation result with the Clark–Ocone formula yields

$$B_T^{-1}I(\eta) = X_0 + \int_0^T \mathbb{E}^{\mathbb{Q}}\{D_t B_T^{-1} I(\eta) | \mathcal{F}_t\} \mathrm{d}W_t$$

$$= X_0 + \int_0^T \sigma_t^*(\Phi_t^1 + \Phi_t^2 + \Phi_t^3)\mathrm{d}W_t,$$

where

$$\Phi_t^1 = \mathbb{E}^{\mathbb{Q}}\{B_T^{-1}\eta I'(\eta)|\mathcal{F}_t\}\Gamma(t, \tilde{P}_t),$$

$$\Phi_t^2 = -\mathbb{E}^{\mathbb{Q}}\left\{ B_T^{-1}\eta I'(\eta) \int_t^T \nabla\lambda_s Y_{t,s} \cdot (\mathrm{d}W_s + \lambda_s \mathrm{d}s) \middle| \mathcal{F}_t \right\},$$

$$\Phi_t^3 = \mathbb{E}^{\mathbb{Q}}\{\big(I(\eta) + \eta I'(\eta)\big)\, Y_{t,T}^* \delta_0 | \mathcal{F}_t\}.$$

Finally, by Theorem 5.7 of (3) we have that if for every $x \geq 0$, the volatility is such that

$$\sigma(t,f)^*\delta_x = \sigma(t,g)^*\delta_x$$

whenever $f(s) = g(s)$ for all $0 \leq s \leq x$, then

$$\mathrm{supp}\{\Phi_t^3\} \subset [0, T-t]$$

for all $t \in [0, T]$.

Acknowledgment The authors would like to thank René Carmona, Thaleia Zariphopoulou, Tomas Björk, Peter Bank, and Gordan Žitković for valuable suggestions. We would also like to thank the anonymous referee for useful comments which have greatly improved the exposition.

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