CHARACTERIZING ATTAINABLE CLAIMS: A NEW PROOF

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Abstract. This short note offers a new proof of the following fact: in a discrete-time arbitrage-free market model, a contingent claim is attainable if and only if its expected value is the same under all equivalent martingale measures. The proof is based on Rogers' proof [9] of the Dalang–Morton–Willinger [1] theorem.

1. Introduction

The purpose of this note is to offer a new proof of the well-known fact that a contingent claim can be replicated by a self-financing trading strategy if and only if it has a unique arbitrage-free price. To formulate this statement mathematically, let $S = (S_t)_{0 \le t \le T}$ be a fixed d-dimensional discrete-time stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to a filtration $(\mathcal{F}_t)_{0 \le t \le T}$, where we assume for simplicity that $\mathcal{F}_0$ is trivial and $\mathcal{F} = \mathcal{F}_T$. Let

$$\mathcal{Q} = \{\mathbb{Q} \sim \mathbb{P} : S \text{ is a } \mathbb{Q}\text{-martingale}\}$$

be the set of equivalent martingale measures. For a d-dimensional predictable process $H = (H_t)_{1 \le t \le T}$ let

$$(H \cdot S)_t = \sum_{s=1}^{t} H_s \cdot \Delta S_s$$

denote the discrete-time stochastic integral, where $a \cdot b$ denotes the standard inner product of vectors $a, b \in \mathbb{R}^d$ and where $\Delta S_t = S_t - S_{t-1}$. Then the well-known fact quoted above amounts to the following theorem:

Theorem 1.1. *Assume that $\mathcal{Q}$ is not empty. For a real-valued random variable X, the following are equivalent:*

1. *$X = (H \cdot S)_T$ a.s. for some predictable process H.*
2. *$\mathbb{E}^{\mathbb{Q}}(X) = 0$ for all $\mathbb{Q} \in \mathcal{Q}$ such that $\mathbb{E}^{\mathbb{Q}}(|X|) < \infty$.*

Theorem 1.1 has been in the folklore of financial mathematics for at least a generation. For instance, Ross [11] in his 1978 paper wrote "… if all admissible operators [i.e. expectations with respect to equivalent martingale measures] lead to the same value, then the return stream must be spanned by existing assets and the resulting value is the proper one for the investment criterion." Indeed, Theorem 1.1 is announced as Corollary (b) of Theorem 2 of Harrison and Kreps's [4] 1979 paper.

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To illustrate the issues involved, let us consider the case when $T = 1$ and the sample space Ω has n elements. Without loss of generality, we will assume $\mathbb{P}\{\omega\} > 0$ for each $\omega \in \Omega$. To switch to linear-algebraic notion, we define a $d \times n$ matrix A by

$$A_{i,j} = \Delta S_1^i(\omega_j).$$

The set

$$\hat{\mathcal{Q}} = \{q \in \mathbb{R}^n : Aq = 0, q_j > 0 \text{ for all } j, \text{ and } q_1 + \ldots + q_n = 1\}$$

corresponds to the set of equivalent martingale measures $\mathcal{Q}$ by the identification $\mathbb{Q}\{\omega_j\} = q_j$. A scalar random variable X corresponds to a vector $x \in \mathbb{R}^n$, so that in this notation Theorem 1.1 says the following are equivalent:

(1) $x = A^{\mathrm{T}} h$ for some $h \in \mathbb{R}^d$.

(2) $q \cdot x = 0$ for all $q \in \hat{\mathcal{Q}}$.

The direction (1) $\Rightarrow$ (2) is easy: if $x = A^{\mathrm{T}} h$ for some $h \in \mathbb{R}^d$, then $q \cdot x = (Aq) \cdot h = 0$ for all $q \in \hat{\mathcal{Q}}$. We now consider the direction (2) $\Rightarrow$ (1). By the orthogonal decomposition $\mathbb{R}^n = \text{Ran } A^{\mathrm{T}} \oplus \text{Ker } A$, it is enough to show that if $Ax = 0$ and $q \cdot x = 0$ for all $q \in \hat{\mathcal{Q}}$ then $x = 0$.

Fix an element $q_0 \in \hat{\mathcal{Q}}$. There exists an $\epsilon \neq 0$ such that

$$q_\epsilon = \frac{\epsilon x + q_0}{\epsilon(x_1 + \ldots + x_n) + 1}$$

is an element of $\hat{\mathcal{Q}}$. Indeed, if ϵ is small enough then

$$(\epsilon x + q_0)_j > 0 \text{ for all } j.$$

Since $q \cdot x = 0$ for all $q \in \hat{\mathcal{Q}}$, we can conclude that

$$0 = q_\epsilon \cdot x = |x|^2 \frac{\epsilon}{\epsilon(x_1 + \ldots + x_n) + 1}$$

and hence $x = 0$ as claimed.

When the sample space Ω is infinite, a naive attempt to extend the above proof fails, even when $T = 1$. Suppose for simplicity that $\Delta S_1 \in L^2$, and let $X \in L^2$ be such that $\mathbb{E}(X \Delta S_1) = 0$ and $\mathbb{E}^{\mathbb{Q}}(X) = 0$ for all $\mathbb{Q} \in \mathcal{Q}$ for which X in integrable. Fix one such measure $\mathbb{Q}_0$ with density

$$Z_0 = \frac{d\mathbb{Q}_0}{d\mathbb{P}}.$$

As before, we can let

$$Z_\epsilon = \frac{\epsilon X + Z_0}{\epsilon \mathbb{E}(X) + 1}.$$

However, we cannot proceed because there is no guarantee that one can choose an $\epsilon \neq 0$ small enough such that $Z_\epsilon > 0$ a.s. Indeed, it may well be the case that $\text{ess inf} Z_0 = 0$, $\text{ess inf} X = -\infty$, and $\text{ess sup} X = \infty$.

The problem with the above approach is that the set $\{Z \in L^2 : Z > 0\}$ is generally not open. Indeed, a functional analytic proof of Theorem 1.1, even in the $T = 1$ case, requires a more careful choice of topologies. The aim of this note is to offer a proof of Theorem 1.1 that by-passes these considerations. Aside from Jensen's inequality, the new proof of this theorem relies on the following representation results for elements of $\mathcal{Q}$:

Theorem 1.2. *The following are equivalent:*

(1) $(S \cdot H)_T = 0$ *a.s. for every predictable* H *such that* $(S \cdot H)_T \geq 0$ *a.s.*
(2) *The set* $\mathcal{Q}$ *is not empty.*
(3) *For every random variable* X *there exist a positive random variable* Z*, predictable processes* H_0 *and* H_1*, positive constants* C_0 *and* C_1*, and (not necessarily distinct) probability measures* $\mathbb{Q}_0$ *and* $\mathbb{Q}_1$ *in* $\mathcal{Q}$ *such that*

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}} = C_0 Z e^{(H_0 \cdot S)_T} \text{ and } \frac{d\mathbb{Q}_1}{d\mathbb{P}} = C_1 Z e^{(H_1 \cdot S)_T + X}.$$

Furthermore, Z *can be chosen such that the random variable*

$$\max\{1, |X|, |(H_0 \cdot S)_T|, |(H_1 \cdot S)_T|\} \max\left\{\frac{d\mathbb{Q}_0}{d\mathbb{P}}, \frac{d\mathbb{Q}_1}{d\mathbb{P}}\right\}$$

is bounded.

The above theorem is a version of the Fundamental Theorem of Asset Pricing. Note that statement (1) above says that the market modelled by S is free of arbitrage opportunities, since any trading strategy H whose return $(H \cdot S)_T$ is always non-negative, must, in fact, always return exactly zero. On the other hand, condition (2) says that there exists at least one equivalent martingale measure. What is significant for us is that condition (2) implies condition (3): that the existence of one equivalent martingale measure automatically implies the existence of two martingale measures with densities given by explicit formulae. The idea of the new proof of Theorem 1.1 is to exploit these formulae when $\mathbb{E}^{\mathbb{Q}}(X) = 0$ for $\mathbb{Q} = \mathbb{Q}_0$ and $\mathbb{Q}_1$ to find a representation of X of the form $X = (H \cdot S)_T$. In fact, we can weaken condition (2) of Theorem 1.1 to

(2′) $\mathbb{E}^{\mathbb{Q}}(X) = 0$ for all $\mathbb{Q} \in \mathcal{Q}$ such that $\max\{1, |X|\}\frac{d\mathbb{Q}}{d\mathbb{P}}$ is is bounded.

since the measures $\mathbb{Q}_0$ and $\mathbb{Q}_1$ verify this boundedness property.

The equivalence (1) $\Leftrightarrow$ (2) in Theorem 1.2 is due to Dalang, Morton, and Willinger [1], though Harrison and Pliska [5] proved the theorem for the case of a finite sample space Ω. The above formulation of the Fundamental Theorem of Asset Pricing is implicit in the papers of Rogers [9, 10]. His proof in the case when $T = 1$, reproduced in Section 3 is simple enough to be included in an advanced undergraduate course on financial mathematics. Indeed, the proof only uses the Bolzano-Weierstrass theorem and the fact that the gradient of a differentiable function vanishes at its maximum. The same proof can be made to work for $T > 1$, but one must take care with the measure-theoretic technicalities.

There are also versions of Theorems 1.1 and 1.2 in continuous time. There is a significant difference between the discrete- and continuous-time theory because one has to be very careful in how to define a class of predictable processes H such that the integral $(H \cdot S)$ is economically meaningful. Indeed, Harrison and Kreps [4] in 1979 already noticed that so-called doubling strategies must not be admissible, otherwise the notion of arbitrage becomes vacuous. In this setting, Jacka [6] proved the appropriate continuous-time version of Theorem 1.1, again confirming the folklore that a claim is attainable if and only if it has the expectation under all equivalent martingale measures. As for the continuous-time version of the Fundamental Theorem of Asset Pricing, Delbaen and Schachermayer [2] proved that a market model has no free lunch with vanishing risk if and only if there exists a probability measure under which the asset prices are σ-martingales. See the recent book of Delbaen and Schachermayer [3] for a survey of these results. Unfortunately, the method of proof detailed below for the discrete-time case does not seem to be directly applicable to the continuous-time case.

The note is arranged as follows: in Section 2, a proof of Theorem 1.1 is presented, including the main contribution of this note – a new, short proof of the implication (2) $\Rightarrow$ (1). In Section 3 we provide the details of Rogers's proof of Theorem 1.2 to keep the note self-contained.

2. A proof of Theorem 1.1

Proof of Theorem 1.1. (1) $\Rightarrow$ (2) Let H be predictable and define a process Y by $Y_t = (H \cdot S)_t$. Suppose the random variable Y_T is integrable with respect to some fixed $\mathbb{Q}$ for which S is a martingale. The case $T = 1$ proceeds exactly as the finite Ω case discussed in the introduction: $\mathbb{E}(Y_1) = H_1 \cdot \mathbb{E}(\Delta S_1) = 0$ since H_1 is not random. However, the case $T > 1$ takes a modicum of care. It is sufficient to show that Y is a martingale since $Y_0 = 0$.

We would like to show that if Y_t is integrable for some $0 < t \leq T$, then $\mathbb{E}(Y_t|\mathcal{F}_{t-1}) = Y_{t-1}$. This would imply Y_{t-1} is integrable because $\mathbb{E}(|Y_{t-1}|) \leq \mathbb{E}(|Y_t|) < \infty$. Since Y_T is integrable by assumption, this would show that Y is a martingale by (backward) induction.

Fix t, suppose Y_t is integrable, and define an increasing sequence of $\mathcal{F}_{t-1}$-measurable events

$$A_n = \{|Y_{t-1}| \leq n, |H_t| \leq n\}$$

on which both Y_{t-1} and H_t are bounded. Note the equality

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{A_n} Y_t|\mathcal{F}_{t-1}) &= \mathbb{E}(\mathbb{1}_{A_n} Y_{t-1}|\mathcal{F}_{t-1}) + \mathbb{E}(\mathbb{1}_{A_n} H_t \cdot \Delta S_t|\mathcal{F}_{t-1}) \\
&= \mathbb{1}_{A_n} Y_{t-1} + \mathbb{1}_{A_n} H_t \cdot \mathbb{E}(\Delta S_t|\mathcal{F}_{t-1}) \\
&= \mathbb{1}_{A_n} Y_{t-1}
\end{aligned}$$

Since Y_t is assumed integrable, we can apply the conditional dominated convergence theorem to send $n \to \infty$ to get $\mathbb{E}(Y_t|\mathcal{F}_{t-1}) = Y_{t-1}$ as desired.

(2) $\Rightarrow$ (1) This the main contribution of this note. Assume $\mathcal{Q}$ is not empty, and that $\mathbb{E}^{\mathbb{Q}}(X) = 0$ for all $\mathbb{Q} \in \mathcal{Q}$ for which X is integrable. Applying Theorem 1.2, we note that

$$X = x + (H \cdot S)_T + \log\left(\frac{d\mathbb{Q}_1}{d\mathbb{Q}_0}\right) \tag{1}$$

for a predictable process $H = H_0 - H_1$, probability measures $\mathbb{Q}_0, \mathbb{Q}_1 \in \mathcal{Q}$, and a constant $x = \log(C_0/C_1)$. We need to show $x = 0$ and $\mathbb{Q}_0 = \mathbb{Q}_1$. Again by Theorem 1.2, we know X and $(H \cdot S)_T$ are integrable with respect to both $\mathbb{Q}_0$ and $\mathbb{Q}_1$. Now by assumption we have $\mathbb{E}^{\mathbb{Q}}(X) = 0$, and by the implication (1) $\Rightarrow$ (2) of this theorem, proven above, we have $\mathbb{E}^{\mathbb{Q}}[(H \cdot S)_T] = 0$ for both $\mathbb{Q} = \mathbb{Q}_0$ and $\mathbb{Q}_1$.

Integrating equation (1) and Jensen's inequality yields

$$\begin{aligned} x &= -\mathbb{E}^{\mathbb{Q}_0} \log\left(\frac{d\mathbb{Q}_1}{d\mathbb{Q}_0}\right) \\ &\geq -\log \mathbb{E}^{\mathbb{Q}_0}\left(\frac{d\mathbb{Q}_1}{d\mathbb{Q}_0}\right) \\ &= 0 \\ &= \log \mathbb{E}^{\mathbb{Q}_1}\left(\frac{d\mathbb{Q}_0}{d\mathbb{Q}_1}\right) \\ &\geq \mathbb{E}^{\mathbb{Q}_1} \log\left(\frac{d\mathbb{Q}_0}{d\mathbb{Q}_1}\right) \\ &= x \end{aligned}$$

so that $x = 0$. But since the logarithmic function is strictly concave, Jensen's inequality holds with equality only if $\frac{d\mathbb{Q}_1}{d\mathbb{Q}_0}$ is constant. Since $\mathbb{Q}_0$ and $\mathbb{Q}_1$ are probability measures, we must conclude $\mathbb{Q}_0 = \mathbb{Q}_1$ as desired. □

3. Rogers' proof of the Fundamental Theorem of Asset Pricing

For completeness, we give a proof of Theorem 1.2. We do this in stages.

Proof of Theorem 1.2 (2) ⇒ (1). Suppose that $\mathcal{Q}$ is not empty, and that $(H \cdot S)_T \geq 0$ a.s. Let $Y = (H \cdot S)$ and fix $\mathbb{Q} \in \mathcal{Q}$.

To see the structure of the argument, we first consider the case where $T = 1$. Since H_1 constant, the random variable $Y_1 = H_1 \cdot \Delta S_1$ is integrable. Since $Y_1 \geq 0$ a.s. but $\mathbb{E}(Y_1) = H_1 \cdot \mathbb{E}(\Delta S_1) = 0$, we must conclude $Y_1 = 0$ a.s.

Now we consider the case where $T > 1$. We will first show that $Y_t \geq 0$ a.s. for all $0 \leq t \leq T$. We proceed by induction. Suppose $Y_t \geq 0$ a.s. for some fixed $0 < t \leq T$, and let

$$A_n = \{|Y_{t-1}| \leq n, |H_t| \leq n\}.$$

As before, we have

$$0 \;\leq\; \mathbb{E}(\mathbb{1}_{A_n} Y_t | \mathcal{F}_{t-1}) = \mathbb{1}_{A_n} Y_{t-1} \to Y_{t-1}$$

as $n \to \infty$, so that $Y_{t-1} \geq 0$ a.s. Since $Y_T \geq 0$ a.s. by assumption, we can conclude $Y_t \geq 0$ a.s. for all $0 \leq t \leq T$ by induction.

We now show $Y_t = 0$ for all $0 \leq t \leq T$. Now suppose $Y_{t-1} = 0$ a.s. for some fixed $0 < t \leq T$. Then

$$0 = \mathbb{1}_{A_n} Y_{t-1} = \mathbb{E}(\mathbb{1}_{A_n} Y_t | \mathcal{F}_{t-1})$$

for every n. But since $\mathbb{1}_{A_n} Y_t \geq 0$ a.s., we must conclude $\mathbb{1}_{A_n} Y_t = 0$ a.s. Letting $n \to \infty$ then implies $Y_t = 0$ a.s. Since $Y_0 = 0$, the (forward) induction is complete. □

Remark 1. It should be noted that neither the proof of the (1) ⇒ (2) direction of Theorem 1.1 nor the proof of the (2) ⇒ (1) direction of Theorem 1.2 given above are especially new and are included only for completeness. Similar ideas can be found, for instance, in Jacod and Shiryaev's proof [7] of the Fundamental Theorem of Asset Pricing.

Since the (3) ⇒ (2) implication of Theorem 1.2 is self-evident, it only remains to prove the (1) ⇒ (3) direction.

Remark 2. We now digress from the formal proof to outline the motivation behind the manipulations that follow. Rogers's proof consists of two steps. The first step is is to look at the classical problem of maximizing the expected utility of terminal wealth. Indeed, let U be a smooth, strictly increasing and concave utility function on $\mathbb{R}$ modelling an investor's aversion to risk, and consider the problem

$$\text{maximize } \mathbb{E}\, U[(H \cdot S)_T]$$

over predictable trading strategies H. If there exists a maximizer H^*, then the following idea is well-known to economists: the marginal utility $U'[(H^* \cdot S)_T]$ of the optimal terminal wealth is proportional to the density of an equivalent martingale measure. To see why, let us suppose for the moment that the sample space Ω is finite to avoid discussing technicalities, although it should be stressed that Ω may be infinite in the proof that follows. Now, for any trading strategy η the function

$$\epsilon \mapsto \mathbb{E}\, U[(H^* \cdot S)_T + \epsilon(\eta \cdot S)_T]$$

is maximized at $\epsilon = 0$, so that

$$\frac{d}{d\epsilon}\, \mathbb{E}\, U[(H^* \cdot S)_T + \epsilon(\eta \cdot S)_T]\big|_{\epsilon=0} = \mathbb{E}\{U'[(H^* \cdot S)_T](\eta \cdot S)_T\} = 0$$

by Fermat's first-order condition for a maximum. Since η was arbitrary, we immediately see that the probability measure $\mathbb{Q}^*$ with density

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \frac{U'[(H^* \cdot S)_T]}{\mathbb{E}\, U'[(H^* \cdot S)_T]}$$

is an equivalent martingale measure.

By modifying the above argument, we can find other formulae for densities of elements of $\mathcal{Q}$. Indeed, given random variables X and Z, where $Z > 0$ a.s., consider the new problem

$$\text{maximize } \mathbb{E}\{ZU[(H \cdot S)_T + X]\}.$$

As before, if there exists a maximizer $H^\star$ to the problem, then the first-order condition yields an equivalent martingale measure $\mathbb{Q}^\star$ with density

$$\frac{d\mathbb{Q}^\star}{d\mathbb{P}} = CZU'[(H^\star \cdot S)_T + rX]$$

for a normalizing constant $C > 0$. It is this modified formulation of the utility maximization idea that will give us the formulae given by condition (3), where we will take, modulo some changes of signs, $U(x) = -e^{-x}$.

The second step of Rogers's proof of Theorem 1.2 consists of showing the no-arbitrage condition implies that a well-chosen utility maximization problem always has a solution.

Proof of Theorem 1.2 (1) $\Rightarrow$ (3). To see the idea of the proof, we first consider the case where $T = 1$. Let $Z = e^{-X^2 - |\Delta S_1|^2}$, and consider two functions F_0 and F_1 on $\mathbb{R}^d$ defined by

$$F_r(h) = \mathbb{E}[Ze^{h\cdot\Delta S_1 + rX}].$$

Note that each F_r is nothing but the moment generating funtion of ΔS_1 with respect to the measure whose density with respect to $\mathbb{P}$ is Ze^{rX}. Each F_r is everywhere finite-valued, thanks to the choice of Z, and hence smooth, and in particular

$$\nabla F_r(h) = \mathbb{E}[Ze^{h\cdot\Delta S_1 + rX}\Delta S_1].$$

Now, if we can show that there exists vectors $h_r \in \mathbb{R}^d$ which minimize F_r then we are done. Indeed, we would have

$$0 = \nabla F_r(h_r) = \mathbb{E}[Ze^{h_r \cdot \Delta S_1 + rX} \Delta S_1]$$

by the first-order condition for the minimizer of a smooth function, and hence the densities

$$\frac{d\mathbb{Q}_r}{d\mathbb{P}} = C_r Z e^{h_r \cdot \Delta S_1 + rX}$$

define equivalent martingale measures, where the $C_r > 0$ are normalizing constants. Note that by the choice of Z, the random variable

$$\max\{1, X, |h_0 \cdot \Delta S_1|, |h_1 \cdot \Delta S_1|\} \max\left\{\frac{d\mathbb{Q}_0}{d\mathbb{P}}, \frac{d\mathbb{Q}_1}{d\mathbb{P}}\right\}$$

is bounded.

We now must show show that the minimizers indeed exist. Fix an $r \in \{0, 1\}$, and drop it from the notation. Now let $(h_n)_n$ be a minimizing sequence, so that

$$F(h_n) \to \inf_h F(h).$$

If $(h_n)_n$ is bounded, then there exists a convergent subsequence, still denoted $(h_n)_n$ so that $h_n \to h^*$. But since F is smooth, we have $F(h_n) \to F(h^*) = \inf_h F(h)$, and hence h^* is our desired minimizer.

It remains to rule out the possibility that $(h_n)_n$ is unbounded. So we now suppose that $(h_n)_n$ is unbounded and aim to show that there exists an arbitrage. Let $\mathcal{V}$ be smallest subspace of $\mathbb{R}^d$ containing the support of the random vector ΔS_1. That is, $\mathcal{V}$ is the orthogonal complement of the subspace

$$\mathcal{U} = \{u \in \mathbb{R}^d : u \cdot \Delta S_1 = 0 \text{ a.s.}\}.$$

In particular, we will use the fact that if $v \in \mathcal{V}$ and $v \neq 0$ then $\mathbb{P}(v \cdot \Delta S_1 = 0) < 1$.

Notice that if $u \in \mathcal{U}$ and $v \in \mathcal{V}$ then $F(u + v) = F(v)$. Hence, we may suppose the minimizing sequence is such that $h_n \in \mathcal{V}$ for all n and $|h_n| \to \infty$.

Letting $\hat{h}_n = h_n/|h_n|$ and noting that the sequence $(\hat{h}_n)_n$ is bounded, we can replace our sequence $(h_n)_n$ with a subsequence such that $\hat{h}_n$ converges to a unit vector $\hat{h} \in \mathcal{V}$. Clearly

$$e^{h_n \cdot \Delta S_1} = (e^{\hat{h}_n \cdot \Delta S_1})^{|h_n|} \to \infty$$

on the event $\{\hat{h} \cdot \Delta S_1 > 0\}$, but since we have

$$\mathbb{E}[\liminf_n Z e^{h_n \cdot \Delta S_1 + rX}] \leq \liminf_n \mathbb{E}[Ze^{h_n \cdot \Delta S_1 + rX}] = \inf_h F(h) \leq F(0) < \infty$$

by Fatou's lemma, we must conclude that $\mathbb{P}(\hat{h} \cdot \Delta S_1 > 0) = 0$ so that $-\hat{h} \cdot \Delta S_1 \geq 0$ a.s. But since $\hat{h} \in \mathcal{V}$ and $\hat{h} \neq 0$, we must have $\mathbb{P}(\hat{h} \cdot \Delta S_1 = 0) < 1$. Hence, $-\hat{h}$ is an arbitrage, contradicting the no-arbitrage assumption (1), completing the proof in the $T = 1$ case.

Now, we consider the case where $T > 1$. The idea is the same as above, but we now have to worry about measurability and integrability. We will build up the processes H_r inductively. Suppose we have constructed the random vectors $H_r(t+1), \ldots, H_r(T)$, for both $r \in \{0, 1\}$, where we begin the procedure at $t = T$ when nothing has been contructed. We now construct $H_r(t)$. As before, we introduce a random factor to ensure integrability. Let

$$Z_t = e^{-X^2 - \sum_{1 \leq s \leq T} |\Delta S_s|^2 - \sum_{t+1 \leq s \leq T} (|H_0(s)|^2 + |H_1(s)|^2)}.$$

Fix t and r, and let P be a regular conditional distribution of the random vector

$$\left(\Delta S_t, X, \sum_{t+1 \leq s \leq T} H_r(s) \cdot \Delta S_s, Z_t \right)$$

given $\mathcal{F}_t$. Now define a function $F : \mathbb{R}^d \times \Omega \to \mathbb{R}$ by

$$\begin{aligned} F(h, \omega) &= \int z e^{rx + h \cdot w + y} dP(w, x, y, z; \omega) \\ &= \mathbb{E}\left[Z_t e^{rX + h \cdot \Delta S_t + \sum_{t+1 \leq s \leq T} H_r(s) \cdot \Delta S_s} | \mathcal{F}_{t-1} \right](\omega). \end{aligned}$$

Notice that for each $\omega \in \Omega$, the function $F(\cdot, \omega)$ is smooth.

For each $\omega \in \Omega$, we will find a $H_r(t, \omega)$ which minimizes $F(\cdot, \omega)$. Let us assume for the moment that this minimizer exists and is $\mathcal{F}_{t-1}$ measurable. The first-order condition for a minimum applies and we have

$$\nabla F(H_r(t)) = \mathbb{E}\left[Z_t e^{rX + \sum_{t \leq s \leq T} H_r(s) \cdot \Delta S_s} \Delta S_t | \mathcal{F}_{t-1} \right] = 0$$

At the end of the induction, we will set

$$\frac{d\mathbb{Q}_r}{d\mathbb{P}} = C_r Z_1 e^{rX + (H_r \cdot S)_T}$$

for an appropriate normalizing constant $C_r > 0$. The above calculation shows that $\mathbb{Q}_r$ is indeed an equivalent martingale measure. By the choice of $Z = Z_1$, it satisfies the announced boundedness property too.

The remaining part of the proof is confirm that an $\mathcal{F}_{t-1}$-measurable minimizer $H_r(t)$ exists. As before, we need to consider a minimizing sequence. However, since $\mathcal{F}_{t-1}$ is generally non-trivial, we need to take a little care with measurability. We consider a sequence of regularized objective functions

$$F_n(h, \omega) = F(h, \omega) + \frac{1}{n} |h|^2.$$

Notice that for all $\omega \in \Omega$, the function $F_n(\cdot, \omega)$ is stictly convex, and that $F_n(h, \omega) \to \infty$ as $|h| \to \infty$. Hence there exists a unique minimizer $H_n(\omega)$. Since

$$F(H_n) \leq F_n(H_n) \leq F_n(h) = F(h) + \frac{1}{n} |h|^2$$

for all $h \in \mathbb{R}^d$, we can first take the lim sup as $n \to \infty$ and then take the infimum over $h \in \mathbb{R}^d$ to conclude that $F(H_n(\omega), \omega) \to \inf_h F(h, \omega)$ for each ω.

This minimizer H_n is $\mathcal{F}_{t-1}$-measurable since

$$\{H_n \in B\} = \bigcup_{p \in Q^d} \bigcap_{q \in Q^d} \{p \in B, q \in B^c, F_n(p) < F_n(q)\}$$

for all open balls $B \subset \mathbb{R}^d$, where Q^d is the countable set of vectors in $\mathbb{R}^d$ with rational coordinates.

Let A be the $\mathcal{F}_{t-1}$-measurable event

$$A = \bigcup_k \bigcap_n \{|H_n| < k\}.$$

that the sequence $(H_n)_n$ is bounded. By Lemma 3.1 below, there an increasing sequence of measurable indices $N_k : A \to \mathbb{N}$ such that $H_{N_k}(\omega)$ converges to some $H(\omega) \in \mathbb{R}^d$ for each $\omega \in A$. Since $F_{N_k}(H_{N_k}) \to F(H)$ pointwise on A, we see that $H = H_r(t)$ will be our desired minimizer as long as we show $\mathbb{P}(A) = 1$.

Now, define the subspace $\mathcal{U}$ orthogonal to the conditional support of ΔS_t given $\mathcal{F}_{t-1}$ by

$$\mathcal{U}(\omega) = \{u \in \mathbb{R}^d : P(\{w \in \mathbb{R}^d : u \cdot w = 0\}, \omega) = 1\}.$$

and let $\mathcal{V}(\omega) = \mathcal{U}(\omega)^\perp$. We see that the minimizer $H_n(\omega)$ is in $\mathcal{V}(\omega)$ for each ω since

$$F_n(u + v) \geq F_n(v)$$

whenever $u \in \mathcal{U}$ and $v \in \mathcal{V}$.

Since $(H_n(\omega))_n$ is unbounded for each $\omega \in A^c$, we can find a measurable sequence of indices $N_k : A^c \to \mathbb{N}$ such that $|H_{N_k}| \to \infty$ on A^c, by taking $N_k = \inf\{n : |H_n| > k\}$. And since the sequence $\hat{H}_{N_k} = H_{N_k}/|H_{N_k}|$ is bounded on A^c, we can find another measurable sequence of indices, again by Lemma 3.1, which we continue to denote by N_k, such that $\hat{H}_{N_k}(\omega)$ converges to $\hat{H}(\omega)$ on A^c. Notice that $\hat{H}(\omega)$ is a unit vector in $\mathcal{V}(\omega)$ for each $\omega \in A^c$. We can extend $\hat{H}$ to a $\mathcal{F}_{t-1}$-measurable random vector by setting $\hat{H} = 0$ on A. As before,

$$e^{H_{N_k} \cdot \Delta S_t} = (e^{\hat{H}_{N_k} \cdot \Delta S_t})^{|H_{N_k}|} \to \infty$$

on the event $A^c \cap \{\hat{H} \cdot \Delta S_t > 0\}$. But by Fatou's lemma we have

$$\mathbb{E}[\liminf_{k\to\infty} e^{H_{N_k} \cdot \Delta S_t} \eta | \mathcal{F}_{t-1}](\omega) \leq \liminf_{k\to\infty} F(H_{N_k}(\omega), \omega) = \inf_h F(h, \omega) \leq F(0, \omega) < \infty$$

for all $\omega \in A^c$ where $\eta = Z_t e^{rX + \sum_{t+1 \leq s \leq T} H_r(s) \cdot \Delta S_s}$. Hence $\mathbb{P}(A^c \cap \{\hat{H} \cdot \Delta S_t > 0\}) = 0$. But this implies $-\mathbb{1}_{A^c} \hat{H} \cdot \Delta S_t \geq 0$ almost surely. Now, using the no-arbitrage condition (1), we can conclude that $\mathbb{1}_{A^c} \hat{H} \cdot \Delta S_t = 0$ almost surely. But this means $\mathbb{1}_{A_c} \hat{H} \in \mathcal{U}$. But since $\hat{H} \in \mathcal{V} = \mathcal{U}^\perp$, we must conclude that $\mathbb{P}(A^c) = 0$, as desired. □

We now state and prove the technical lemma used in the proof above. It is, in a sense, a measurable version of the Bolzano–Weierstrass theorem. It appears in the paper of Kabanov and Stricker [8] who thank Engelbert and Weizsäcker for suggesting it.

Lemma 3.1. *Consider a sequence of measurable functions $\xi_n : B \to \mathbb{R}^d$ such that $\sup_n |\xi_n(\omega)| < \infty$ for all $\omega \in B$. Then there exists an increasing sequence of measurable indices $N_k : B \to \mathbb{N}$ such that $(\xi_{N_k}(\omega))_k$ converges for each $\omega \in B$.*

Proof. We consider the case $d = 1$ first. Let

$$N_k = \inf\{h \geq 1 : \xi_h \geq \limsup_{n\to\infty} \xi_n - 1/k\}.$$

It is easy to see that the N_k are measurable, so that the ξ_{N_k} are also measurable and $\xi_{N_k} \to \limsup_{n\to\infty} \xi_n$ pointwise.

For the case $d > 1$, we simply work component-wise. First let $N^0_k = k$ for all $k \geq 1$, and define recursively, for $1 \leq i \leq d$,

$$N^i_k = \inf\{h \geq 1 : \xi^i_{N^{i-1}_h} \geq \limsup_{n\to\infty} \xi^i_{N^{i-1}_n} - 1/k\}.$$

Now take $N_k = N^d_k$. □

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References

[1] R.C. Dalang, A. Morton, and W. Willinger. (1990) Equivalent martingale measures and no-arbitrage in stochastic securities markets. *Stochastics and Stochastics Reports* **29**: 185–201.

[2] F. Delbaen and W. Schachermayer. (1998) The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen* **312**: 215–250.

[3] F. Delbaen and W. Schachermayer. (2006) *The Mathematics of Arbitrage*. Springer Finance.

[4] J.M. Harrison and D.M. Kreps. (1979) Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* **20**: 381–408.

[5] J.M. Harrison and S.P. Pliska. (1981) Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications* **11**: 215–260.

[6] S.D. Jacka. (1992) A martingale representation result and an application to incomplete financial markets. *Mathematical Finance* **2**: 23–34.

[7] J. Jacod and A.N. Shiryaev. (1998) Local martingales and the fundamental asset pricing theorems in the discrete-time case. *Finance and Stochastics* 2: 259-273.

[8] Y. Kabanov and C. Stricker. (2001) A teacher's note on no-arbitrage criteria. *Séminaire de probabilités de Strasbourg* **35**: 149–152.

[9] L.C.G. Rogers. (1994) Equivalent martingale measures and no-arbitrage. *Stochastics and Stochastics Reports* **51**: 41–49.

[10] L.C.G. Rogers. (1998) The origins of risk-neutral pricing and the Black–Scholes formula. In *Handbook of Risk Management and Analysis*, ed. C. O. Alexander, Wiley, Chichester.

[11] S.A. Ross. (1978) A simple approach to the valuation of risky streams. *The Journal of Business* **51**(3): 453–475.

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