UNIFORM BOUNDS FOR BLACK–SCHOLES IMPLIRED VOLATILITY

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Abstract. In this note, Black–Scholes implied volatility is expressed in terms of various optimisation problems. From these representations, upper and lower bounds are derived which hold uniformly across moneyness and call price. Various symmetries of the Black–Scholes formula are exploited to derive new bounds from old. These bounds are used to reprove asymptotic formulae for implied volatility at extreme strikes and/or maturities. Finally, a curious characterisation of log-concave distributions on the real line is derived, generalising the main optimisation-based representation.

1. Introduction

We define the Black–Scholes call price function $C_{BS} : \mathbb{R} \times [0, \infty) \to [0, 1)$ by the formula

$$C_{BS}(k, y) = \int_{-\infty}^{\infty} (e^{yz-y^2/2} - e^k)^+ \phi(z) dz$$

$$= \begin{cases} \Phi\left( -\frac{k}{y} + \frac{y}{2}\right) - e^k \Phi\left( -\frac{k}{y} - \frac{y}{2}\right) & \text{if } y > 0 \\ (1 - e^k)^+ & \text{if } y = 0, \end{cases}$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$ is the standard normal density and $\Phi(x) = \int_{-\infty}^{x} \phi(z) dz$ is its distribution function. As is well known, the financial significance of the function $C_{BS}$ is that, within the context of the Black–Scholes model [4], the minimal replication cost of a European call option with strike $K$ and maturity $T$ written on a stock with initial price $S_0$ is given by

replication cost $= S_0 e^{-\delta T} C_{BS} \left[ \log \left( \frac{K e^{-r T}}{S_0 e^{-\delta T}} \right), \sigma \sqrt{T} \right]$.

where $\delta$ is the dividend rate, $r$ is the interest rate and $\sigma$ is the volatility of the stock. Therefore, in the definition of $C_{BS}(k, y)$, the first argument $k$ plays the role of log-moneyness of the option and the second argument $y$ is the total standard deviation of the terminal log stock price.

Of the six parameters appearing in the Black–Scholes formula for the replication cost, five are readily observed in the market. Indeed, the strike $K$ and maturity date $T$ are specified by the option contract, and the initial stock price $S_0$ is quoted. The interest rate is the yield of a zero-coupon bond $B_{0,T}$ with maturity $T$ and unit face value, and can be computed from the initial bond price via $B_{0,T} = e^{-rT}$. Similarly, the dividend rate can computed from the stock’s initial time-$T$ forward price $F_{0,T} = S_0 e^{(r-\delta)T}$.

As suggested by Latané & Rendleman [16] in 1976, the remaining parameter, the volatility $\sigma$, can also be inferred from the market, assuming that the call has a quoted price $C_{\text{quoted}}$.

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Indeed, note that for fixed $k$, the map $C_{BS}(k, \cdot)$ is strictly increasing and continuous, so we can define the inverse function

$$Y_{BS}(k, \cdot) : [(1 - e^k)^+, 1) \to [0, \infty)$$

by

$$y = Y_{BS}(k, c) \iff C_{BS}(k, y) = c.$$ 

The implied volatility of the call option is then defined to be

$$\sigma^{\text{implied}} = \frac{1}{\sqrt{T}} Y_{BS} \left[ \log \left( \frac{Ke^{-rT}}{S_0 e^{-\delta T}} \right), \frac{C^{\text{quoted}}}{S_0 e^{-\delta T}} \right].$$

Because of its financial significance, the function $Y_{BS}$ has been the subject of much interest. For instance, approximations for $Y_{BS}$ can be found in several papers [5, 6, 18, 21]. Unfortunately, there seems to be only one case where the function $Y_{BS}$ can be expressed explicitly in terms of elementary functions: when $k = 0$ we have

$$C_{BS}(0, y) = 2 \Phi \left( \frac{y}{2} \right) - 1$$

$$= 1 - 2 \Phi \left( -\frac{y}{2} \right)$$

and hence

$$Y_{BS}(0, c) = 2 \Phi^{-1} \left( \frac{1 + c}{2} \right)$$

$$= -2 \Phi^{-1} \left( \frac{1 - c}{2} \right).$$

The main contribution of this article is to provide bounds on the quantity $Y_{BS}(k, c)$ in terms of elementary functions of $(k, c)$. As an example, in Proposition 4.3 below we will see that

(1) $$Y_{BS}(k, c) \leq -2 \Phi^{-1} \left( \frac{1 - c}{1 + e^k} \right)$$

for every $(k, c)$ such that $(1 - e^k) \leq c < 1$.

We list here two possible applications of such bounds. When $k \neq 0$, the function $Y_{BS}$ can be evaluated numerically. A simple way to do so is to implement the bisection method for finding the root of the map $y \mapsto C_{BS}(k, y) - c$. That is to say, for fixed $(k, c)$ pick two points $\ell < u$ such that $C_{BS}(k, \ell) < c$ and $C_{BS}(k, u) > c$. By the intermediate value theorem, we know that the root is in the interval $(\ell, u)$. We then let $m = \frac{1}{2}(\ell + u)$ be the midpoint. If $C_{BS}(k, m) > c$ we know that the root $Y_{BS}(k, c)$ is in the interval $(\ell, m)$, in which case we relabel $m$ as $u$. Similarly, if $C_{BS}(k, m) < c$ we relabel $m$ as $\ell$. This process is repeated until $|C_{BS}(k, m) - c| < \varepsilon$, where $\varepsilon > 0$ is a given tolerance level whereupon we declare $Y_{BS}(k, c) \approx m$. (We note here that a more sophisticated idea is apply the Newton–Raphson method as suggested by Manaster & Koehler [20] in 1982. We will return to this in Section 3.)

In order to implement the bisection method, we need a lower bound $\ell$ and upper bound $u$ to initialise the algorithm. However, aside from the obvious lower bound $\ell = 0$, there do not seem to be many well-known explicit upper and lower bounds on the quantity $Y_{BS}(k, c)$.
which hold uniformly in \((k,c)\). This note provides such bounds, and indeed, equation (1) is an example.

We now consider another application of our bounds. Consider a market model with a zero-coupon bond with maturity date \(T\) whose time-\(t\) price is \(B_{t,T}\) and a stock with time \(t\)-price \(S_t\). Suppose the initial price of a call option with strike \(K\) and maturity \(T\) is given by

\[
C_{\text{quoted}} = B_{0,T} \mathbb{E}^T[ (S_T - K) ]
\]

where the expectation is under a fixed \(T\)-forward measure. Further, suppose the stock’s initial time-\(T\) forward price is given by

\[
F_{0,T} = \mathbb{E}^T[S_T].
\]

(If the stock pays no dividend, arbitrage considerations would imply \(F_{0,T} = S_0/B_{0,T}\). However, we do not need this formula here so the stock is allowed to pay dividends in this discussion.) Now, equation (1) implies that the implied volatility is bounded by

\[
\sigma_{\text{implied}} = \frac{1}{\sqrt{T}} Y_{BS} \left[ \log \left( \frac{K}{F_{0,T}} \right), \frac{C_{\text{quoted}}}{F_{0,T} B_{0,T}} \right] \\
\leq -\frac{2}{\sqrt{T}} \Phi^{-1} \left( \frac{\mathbb{E}^T[S_T \wedge K]}{\mathbb{E}^T[S_T] + K} \right).
\]

Note that the above bound is the composition of two ingredients: the model-dependent formulae for the quantities \(C_{\text{quote}}\) and \(F_{0,T}\), and a uniform and model-independent bound on the function \(Y_{BS}\).

There has been much recent interest in implied volatility asymptotics. See for instance the papers [2, 3, 7, 9, 10, 11, 12, 17, 22, 24] for asymptotic formulae which depend on minimal model data, such as distribution function or the moment generating function of the returns of the underlying stock. Paralleling the discussion above, such asymptotic formulae can be seen as compositions of two limits: first, the asymptotic shape of the call surface as predicted by the model at, for instance, extreme strikes and/or maturities; and second, asymptotics of the model-independent function \(Y_{BS}\). The uniform bounds on \(Y_{BS}\) that are presented in this note are used to provide short, new proofs of these second model-independent asymptotic formulae.

In their long survey article, Andersen & Lipton [1] warn that many of the asymptotic implied volatility formulae that have appeared in recent years may not be applicable in practice, since typical market parameters are usually not in the range of validity of any of the proposed asymptotic regimes. Our new bounds on the function \(Y_{BS}\) are uniform, and hence side-step the critique of Andersen & Lipton.

The rest of the note is organised as follows. In Section 2 we discuss various symmetries of the Black–Scholes pricing function \(C_{BS}\). These symmetries will be used repeated throughout the remainder of the note. In Section 3 the Black–Scholes implied total standard deviation function \(Y_{BS}\) is represented as the value function of several optimisation problems. These results constitute the main contribution of this note, since they allow \(Y_{BS}\) to be bounded arbitrarily well from above and below by choosing suitable controls to insert into the respective objective functions. In Section 4 these bounds are used to reprove some known asymptotic formulae. As a by-product, we derive formulae which have the same asymptotic behaviour as the known formulae, but are guaranteed to bound \(Y_{BS}\) either from above or below. Finally,
in Section 5, we revisit the new formula in the context of convex analysis. In particular, we use convex duality to give a characterisation of log-concave distributions.

2. Put-call and close-far symmetries

The Black–Scholes call price function $C_{BS}$ contains a certain amount of symmetry. In order to streamline the presentation of our bounds, we begin with an exploration of two of these symmetries.

To treat the two cases $k \geq 0$ and $k < 0$ as efficiently as possible, we begin with an observation. Suppose $c$ is the normalised price of a call option with log-moneyness $k$. Then by the usual put-call parity formula, the corresponding normalised price of a put option with the same log-moneyness is

$$p = c + e^k - 1.$$  

Now if $c = C_{BS}(k, y)$ is for some $y > 0$, then we have

$$p = C_{BS}(k, y) + e^k - 1$$

$$= e^k \Phi \left( \frac{k}{y} + \frac{y}{2} \right) - \Phi \left( \frac{k}{y} - \frac{y}{2} \right)$$

$$= e^k C_{BS}(-k, y).$$

The above calculation is the well-known Black–Scholes put-call symmetry formula. We have just proven the following result:

**Proposition 2.1.** For any $k \in \mathbb{R}$ and $c \in [(1 - e^k)^+, 1)$ we have

$$Y_{BS}(k, c) = Y_{BS}(-k, e^{-k}c + 1 - e^{-k}).$$

One conclusion of proposition 2.1 is that it is sufficient to study the function $Y_{BS}(k, \cdot)$ only in the case $k \geq 0$. Indeed, to study the case $k < 0$ one simply applies the above put-call symmetry formula.

We now come to another, less well-known, symmetry of the Black–Scholes formula. While put-call symmetry involves replacing the log-moneyness $k$ with $-k$, the symmetry discussed here involves replacing the total standard deviation $y$ with $2|k|/y$. By put-call symmetry, we can confine our discussion to the case $k > 0$.

**Proposition 2.2.** For all $k > 0$ and $0 < c < 1$, let

$$\hat{C}(k, c) = 1 - \int_0^c \frac{2k}{Y_{BS}(k, u)^2} du.$$  

Then $\hat{C}(k, c) > 0$ and we have

$$Y_{BS}(k, c) = \frac{2k}{Y_{BS}(k, \hat{C}(k, c))}.$$  

Figure 1 is shows the graph of $c \mapsto \hat{C}(k, c)$ when $k = 0.2$.

**Proof.** We must prove that

$$\hat{C}(k, c) = C_{BS} \left( k, \frac{2k}{Y_{BS}(k, c)} \right).$$  

or equivalently

\[ \hat{C}(k, C_{BS}(k, y)) = C_{BS}\left(k, \frac{2k}{y}\right), \]

The above identity can be verified by differentiating both sides with respect to \( y \), and using the Black–Scholes vega formula: for \( k > 0 \), we have

\[ C_{BS}(k, y) = \int_{0}^{y} \phi\left(-k/x + x/2\right) dx. \]

\[ \square \]

Remark 2.3. Consider a stock which pays no dividend \( \delta = 0 \) in a market with zero risk free interest rate \( r = 0 \). Recall that for a call option very close to maturity, that is when \( T \downarrow 0 \), standard arbitrage-free models would predict that the option price is very close to \( (S_0 - K)^+ \) and hence the implied total standard deviation \( \sqrt{T}\sigma_{\text{implied}} \) is very small. Conversely, when the open is very far from maturity when \( T \uparrow \infty \), the call price in many models is close to \( S_0 \) in which case the total standard deviation \( \sqrt{T}\sigma_{\text{implied}} \) is very large. Proposition 2.2 then could be considered a symmetry relation between short-dated and long-dated options.

We conclude this section with some easy observations which we will use later.

**Proposition 2.4.** For all \( k > 0 \), the function \( \hat{C}(k, \cdot) \) is convex and satisfies the functional equation

\[ \hat{C}(k, \hat{C}(k, c)) = c \]

holds for all \( 0 < c < 1 \).

**Proof.** It is easy to see that \( Y_{BS}(k, \cdot) \) is strictly increasing. That \( \hat{C}(k, \cdot) \) is convex follows from the fact that its gradient \(-2k/Y_{BS}(k, \cdot)^2\) is increasing.

\[ \]
That the functional equation is proven by noting
\[ Y_{BS}(k, c) = \frac{2k}{Y_{BS}(k, \hat{C}(k, \hat{c}))} = Y_{BS}(k, \hat{C}(k, \hat{c}(k, c))) \]
and using the fact that that $Y_{BS}(k, \cdot)$ is strictly increasing. \hfill \blacksquare

**Proposition 2.5.** For $k > 0$, let
\[ J(k, c) = \int_0^c \frac{du}{Y_{BS}(k, u)} du. \]
Then
\[ J(k, c) + J(k, \hat{c}) = J(k, 1) \]
where $\hat{c} = \hat{C}(k, c)$.

**Proof.** By setting $c = C_{BS}(k, y)$ and hence $\hat{c} = C_{BS}(k, 2k/y)$, the identity can be proven by computing the derivative with respect to $y$ of the left-hand side, and note that it is vanishes identically. \hfill \blacksquare

**Remark 2.6.** By changing variables, we have the identities
\[ J(k, 1) = \int_0^\infty \frac{\phi(-k/y + y/2)}{y} dy \]
\[ = \int_{-\infty}^\infty \frac{\phi(x)}{\sqrt{x^2 + 2k}} dx \]
\[ = \frac{e^{k/2}}{\sqrt{2\pi}} K_0(k/2) \]
where $K_0$ is a modified Bessel function. See [8].

3. **Various optimisation problems**

This section contains one of the main results of this note, formulae for the function $Y_{BS}$ in terms of various optimisation problems. The first results is that $Y_{BS}(k, c)$ can be calculated by solving a minimisation problem. In particular, we can use this formula to find an upper bound simply by evaluating the objective function at a feasible control.

**Theorem 3.1.** For all $k \in \mathbb{R}$ and $(1 - e^k)^+ \leq c < 1$ we have
\[ Y_{BS}(k, c) = \inf_{d_1 \in \mathbb{R}^+} [d_1 - \Phi^{-1}(e^{-k}(\Phi(d_1) - c))] \]
\[ = \inf_{d_2 \in \mathbb{R}^+} [\Phi^{-1}(c + e^k \Phi(d_2)) - d_2] \]

Furthermore, if $c > (1 - e^k)^+$, then the two infima are attained at
\[ d_1^* = -\frac{k}{y} + \frac{y}{2}, \]
\[ d_2^* = -\frac{k}{y} - \frac{y}{2} \]
where $y = Y_{BS}(k, c)$. 
Remark 3.2. We are using the convention that $\Phi^{-1}(u) = +\infty$ for $u \geq 1$ and $\Phi^{-1}(u) = -\infty$ for $u \leq 0$.

The following proof is due to Pieter-Jan De Smet [23], simplifying the proof in an early version of this paper. The idea is essentially that the inequality

$$(X - K)^+ \geq (X - K)1_{[H,\infty)}(X)$$

holds for all $X, K, H \geq 0$ with equality if and only if $H = K$.

Proof. Fix $k \in \mathbb{R}$ and $(1 - e^k)^+ \leq c < 1$ and let $y \geq 0$ be such that $C_{BS}(k, y) = c$. Note that for any $d_2 \in \mathbb{R}$ we have

$$c = \int_{-\infty}^\infty (e^{yz - y^2/2} - e^k)^+ \phi(z) \, dz$$

$$\geq \int_{-d_2}^\infty (e^{yz - y^2/2} - e^k)^+ \phi(z) \, dz$$

$$\geq \int_{-d_2}^\infty (e^{yz - y^2/2} - e^k) \phi(z) \, dz$$

$$= \Phi(d_2 + y) - e^k \Phi(d_2).$$

There is equality from the first to second line only if $-d_2 \leq k/y + y/2$, and there is equality from the second to the third line only if $-d_2 \geq k/y + y/2$. Rearranging then yields

$$y \leq \Phi^{-1}(c + e^k \Phi(d_2)) - d_2.$$

Let $d_2 = \Phi^{-1}(e^{-k}(\Phi(d_1) - c))$ in the above inequality to obtain the first expression. □

Let

(2) \quad $H_1(d; k, c) = d - \Phi^{-1}(e^{-k}(\Phi(d) - c))$

and

(3) \quad $H_2(d; k, c) = \Phi^{-1}(c + e^k \Phi(d)) - d,$

and note that

$H_1(d; k, c) = H_2(-d; -k, e^{-k}c + 1 - e^{-k})$

in line with put-call symmetry. We use this notation to compute $Y_{BS}(k, c)$ in terms of a maximisation problem. This representation can be used, in principle, to find lower bounds.

Theorem 3.3. Let $C$ be the space of continuous functions on $[0, 1]$. For $k > 0$ and $0 < c < 1$, we have

$$Y_{BS}(k, c) = \sup_{D \in C, d \in \mathbb{R}} \frac{2k}{H_i(d; k, 1 - \int_0^c \frac{2k}{H_i(D(u), k, u)} \, du)}$$

for any $i, j \in \{1, 2\}$. 7
Proof. By Theorem 3.1 we have $Y_{BS}(k, u) \leq H_j(d; k, u)$ for all $d$, and since $Y_{BS}(k, \cdot)$ is increasing, we have for any $D \in \mathcal{C}$ that

$$Y_{BS}(k, \hat{C}(k, c)) = Y_{BS} \left( k, 1 - \int_0^c \frac{2k}{Y_{BS}(k, u)^2} du \right) \leq Y_{BS} \left( k, 1 - \int_0^c \frac{2k}{H_j(D(u); k, u)^2} du \right) \leq H_i \left( d; k, 1 - \int_0^c \frac{2k}{H_j(D(u); k, u)^2} du \right).$$

The conclusion follows from Proposition 2.2. \[\square\]

In light of Proposition 2.2 we now give a representation of $\hat{C}$ in terms of a minimisation problem. We restrict attention to $k > 0$ with no real loss thanks to put-call symmetry.

**Proposition 3.4.** For $k > 0$ and $0 < c < 1$ we have

$$\hat{C}(k, c) = \sup_{y \geq 0} \left[ C_{BS} \left( k, \frac{2k}{y} \right) - \frac{2k}{y^2} (c - C_{BS}(k, y)) \right].$$

Proof. Recall that by Proposition 2.4 that $\hat{C}(k, \cdot)$ is convex. Hence

$$\hat{C}(k, c) - \hat{C}(k, c^*) \geq - \frac{2k}{Y_{BS}(k, c^*)^2} (c - c^*).$$

for any $c, c^* \in (0, 1)$. Letting $y = Y_{BS}(k, c^*)$ we have

$$\hat{C}(k, c) \geq C_{BS} \left( k, \frac{2k}{y} \right) - \frac{2k}{y^2} (c - C_{BS}(k, y))$$

as claimed. \[\square\]

Of course, there are other representations of $Y_{BS}$ in terms of an optimisation problems. For instance, we have

$$Y_{BS}(k, c) = \inf \{ y \geq 0 : C_{BS}(k, y) \geq c \} = \sup \{ y \geq 0 : C_{BS}(k, y) \leq c \}. $$

Indeed, this simple observation underlies the bisection method discussed in the introduction.

We conclude this section with a slightly more interesting representation. It be can used to find upper and lower bounds of $Y_{BS}(k, c)$, at least in principle. However, in practice it is not clear how to choose candidate controls, so we do not explore this idea in the sequel. This result is due to Manaster & Koehler [20], and is motivated by the Newton–Raphson method for computing implied volatility numerically.

**Proposition 3.5 (Manaster & Koehler).** Fix $k \geq 0$ and $0 \leq c < 1$. If $c \leq C_{BS}(k, \sqrt{2k}) = 1/2 - e^k \Phi(-\sqrt{2k})$ then

$$Y_{BS}(k, c) = \inf_{0 \leq y \leq \sqrt{2k}} \left[ y + \frac{c - C_{BS}(k, y)}{\phi(-k/y + y/2)} \right].$$
Otherwise, if \( c \geq 1/2 - \varepsilon k \Phi(-\sqrt{2k}) \) then
\[
Y_{BS}(k, c) = \sup_{y \geq \sqrt{2k}} \left[ y + \frac{c - C_{BS}(k, y)}{\phi(-k/y + y/2)} \right].
\]

**Proof.** The restriction of \( C_{BS}(k, \cdot) \) to \( 0, \sqrt{2k} \) is convex, as can be confirmed by differentiation. Hence, by the Black–Scholes vega formula, we have
\[
C_{BS}(k, y^*) - C_{BS}(k, y) \geq \phi(-k/y + k/y)(y^* - y)
\]
for any \( y, y^* \in [0, \sqrt{2k}] \). Fixing \( y^* \) and letting \( c = C_{BS}(k, y^*) \) we have proven
\[
y^* \leq y + \frac{c - C_{BS}(k, y)}{\phi(-k/y + y/2)}
\]
as desired. Similarly, since the restriction of \( C_{BS}(k, \cdot) \) to \( [\sqrt{2k}, \infty) \) is concave the second conclusion follows. \( \square \)

### 4. Uniform bounds and asymptotics

In this section, we will offer quick proofs of some asymptotic formulae for the function \( Y_{BS} \). These formulae already appear in the literature, but important novelty here is that we will derive bounds on the function \( Y_{BS} \) which hold uniformly, not just asymptotically. To obtain upper bounds in most cases, we simply choose a convenient \( d_1 \) or \( d_2 \) to plug into Theorem 3.1. Note that the proposed upper bound is close to the true value of \( Y_{BS}(c, k) \) when, for instance, the proposed value of \( d_1 \) is close to the minimiser \( d_1^* = -k/y + y/2 \). In principle, lower bounds could be found by choosing convenient controls into Theorem 3.3. However, in practice, we have found other arguments, while lacking the same unifying principle, which do have the advantage of being simple. In the proofs that follow, we usually only consider the \( k \geq 0 \) case, as the \( k < 0 \) case follows directly from Proposition 2.1.

Before we begin, we need a lemma regarding the asymptotic behaviour of the standard normal quantile function \( \Phi^{-1} \).

**Lemma 4.1.** As \( \varepsilon \downarrow 0 \) we have
\[
[\Phi^{-1}(\varepsilon)]^2 = -2 \log \varepsilon + O(\log(-\log \varepsilon)).
\]
In particular, we have
\[
\Phi^{-1}(\varepsilon) = -\sqrt{-2 \log \varepsilon} + O\left(\frac{\log(-\log \varepsilon)}{\sqrt{-\log \varepsilon}}\right)
\]

**Proof.** Let \( \varepsilon = \Phi(-x) \) for large \( x > 0 \) and let
\[
R(x) = \frac{\Phi(-x)x}{\phi(x)}.
\]
In this notation we have the identity
\[
\log \Phi(-x) = -x^2/2 - \log(\sqrt{2\pi}x) + \log R(x).
\]
Since it is well known that \( R(x) \to 1 \) as \( x \to \infty \) we have
\[
\frac{\log \Phi(-x)}{x^2} \to -1/2
\]
or equivalently
\[ \frac{[\Phi^{-1}(\varepsilon)]^2}{\log \varepsilon} \to -2. \]
Plugging in this limit into the identity yields the first conclusion, and Taylor’s theorem yields the second.

The first example comes from [24]. This asymptotic formula considers the behaviour of \( Y_{BS} \) when \( c \) is close to its upper bound of 1. This result is useful in studying implied volatility at very long maturities, when the strike is fixed.

**Theorem 4.2.** For fixed \( k \in \mathbb{R} \), we have
\[ Y_{BS}(k, c) = \sqrt{-8 \log(1 - c)} + O\left(\frac{\log[-\log(1 - c)]}{\sqrt{-\log(1 - c)}}\right) \]
as \( c \uparrow 1 \).

The proof of the above theorem relies the following simple bounds which hold uniformly in \((c, k)\).

**Proposition 4.3.** Fix \( k \in \mathbb{R} \) and \((1 - \varepsilon^k)^+ \leq c < 1\). For \( k \geq 0 \) we have
\[ -2\Phi^{-1}\left(\frac{1 - c}{2}\right) \leq Y_{BS}(k, c) \leq -2\Phi^{-1}\left(\frac{1 - c}{1 + \varepsilon^k}\right) \]
and for \( k < 0 \) we have
\[ -2\Phi^{-1}\left(\frac{1 - c}{2e^k}\right) \leq Y_{BS}(k, c) \leq -2\Phi^{-1}\left(\frac{1 - c}{1 + e^k}\right). \]

**Proof.** For the upper bound, let \( d_2 = \Phi^{-1}\left(\frac{1 - c}{1 + \varepsilon^k}\right) \) in Theorem 3.1.

For the lower bound, let \( y = Y_{BS}(k, c) \). Note that \( C_{BS}(\cdot, y) \) is decreasing and hence
\[ 1 - 2\Phi(-y/2) = C_{BS}(0, y) \geq C_{BS}(k, y) = c \]
when \( k \geq 0 \). In the case when \( k < 0 \), note that
\[ \frac{1 - e^{-k}p}{1 + e^{-k}} = \frac{1 - c}{1 + e^k} \]
and that
\[ \frac{1 - e^{-k}p}{2} = \frac{1 - c}{2e^k}. \]
Now appeal to the put-call parity formula of Proposition 2.1.

**Proof of Theorem 4.2.** By Proposition 4.3 and Lemma 4.1, we have
\[ Y_{BS}(k, c) \leq -2\Phi^{-1}\left(\frac{1 - c}{1 + e^k}\right) \]
\[ = \sqrt{-8 \log(1 - c)} + O\left(\frac{\log[-\log(1 - c)]}{\sqrt{-\log(1 - c)}}\right) \]
where we have used the fact that for fixed \( k \) we have
\[
\sqrt{-2 \log \left( \frac{1 - c}{1 + e^k} \right)} = \sqrt{-2 \log(1 - c)} + O\left( \frac{1}{\sqrt{-\log(1 - c)}} \right)
\]
as \( c \uparrow 1 \).

Similarly, by Proposition 4.3, we have for \( k \geq 0 \) that
\[
Y_{BS}(k, c) \geq -2 \Phi^{-1} \left( \frac{1 - c}{2} \right)
\]
\[
= \sqrt{-8 \log(1 - c)} + O \left( \frac{\log[-\log(1 - c)]}{\sqrt{-\log(1 - c)}} \right).
\]
The \( k < 0 \) is identical.

The next example we consider in this section is due to Roper & Rutkowski [22] and deals with the case where \( c \) is close to its lower bound \((1 - e^k) + \). In particular, this regime is useful for studying the implied volatility smile of options very near maturity.
Theorem 4.4 (Roper & Rutkowski). If $k > 0$ then
\[ Y_{BS}(k, c) = \frac{k}{\sqrt{-2 \log c}} + O\left(\frac{\log(-\log c)}{(-\log c)^{3/2}}\right) \]
as $c \downarrow 0$. If $k < 0$ then
\[ Y_{BS}(k, c) = \frac{-k}{\sqrt{-2 \log p}} + O\left(\frac{\log(-\log p)}{(-\log p)^{3/2}}\right) \]
as $c \downarrow 1 - e^k$, where $p = c + e^k - 1$.

As always, we will prove the asymptotic result by finding uniform bounds. As discussed in Section 2, we can reuse of the bounds which are tight when $c$ is close to 1 by first bounding the function $\hat{C}$.

Proposition 4.5. For $k > 0$ and $0 < c < 1$, we have
\[ 1 - c L(k, c) \leq \hat{C}(k, c) \leq 1 - c \]
where
\[ L(k, c) = \frac{2}{k} \left[ \Phi^{-1}\left(\frac{c}{1 + e^k}\right)^2 + 2\right]. \]

Proof. For the upper bound, simply note that
\[ C_{BS}(k, y) + C_{BS}(k, 2k/y) = 1 - 2e^k\Phi(-k/y - y/2) \leq 1. \]

Now
\[ \hat{C}(k, c) = 1 - \int_0^c Y_{BS}(k, u)^2 du \]
\[ = 1 - \int_0^c \frac{2k}{Y_{BS}(k, u)^2} du \]
\[ \geq 1 - \int_0^c \frac{2k}{Y_{BS}(k, 1 - u)^2} du \]
by two applications of Proposition 2.2 and the upper bound.

Now, we appeal to the upper bound in Proposition 4.3 to conclude that
\[ \hat{C}(k, c) \geq 1 - \frac{2}{k} \int_0^c \Phi^{-1}\left(\frac{u}{1 + e^k}\right)^2 du \]
\[ = 1 - \frac{2(1 + e^k)}{k} \int_{-\Phi^{-1}\left(\frac{c}{1 + e^k}\right)}^{\infty} x^2 \phi(x) dx. \]

To complete the proof, note that the bound
\[ \int_A^{\infty} x^2 \phi(x) dx = A\phi(A) + \Phi(-A) \leq (A^2 + 2)\Phi(-A) \]
which holds for all $A \geq 0$. \qed
We now prove an inequality which provides an easy way to convert bounds which are good when \( c \uparrow 1 \) into bounds which are good when \( c \downarrow 0 \).

**Proposition 4.6.** Fix \( k \geq 0 \) and \( 0 < c < 1 \). Then

\[
\frac{2k}{Y_{BS}(k, 1 - c)} \leq Y_{BS}(k, c) \leq \frac{2k}{Y_{BS}(k, 1 - c L(k, c))}.
\]

where \( L(k, c) \) is defined by equation (4). In particular, we have

\[
Y_{BS}(k, c) \geq \frac{k}{-\Phi^{-1}\left(\frac{c}{1+c^k}\right)}
\]

and if \( c L(k, c) \leq 1 \) we have

\[
Y_{BS}(k, c) \leq \frac{k}{-\Phi^{-1}\left(\frac{c}{1+c^k}\right)}.
\]

**Proof.** The first claim follows from the fact that \( Y_{BS}(k, \cdot) \) is increasing and Proposition 2.2. The second set of claims follow from the bounds in Proposition 4.3. \( \square \)

**Remark 4.7.** The inequality

\[
Y_{BS}(k, c)Y_{BS}(k, 1 - c) \geq 2k
\]

which holds for all \( k > 0 \) and \( 0 < c < 1 \), has an appealing symmetry!

**Proof of Theorem 4.4.** First fix \( k > 0 \). Using the second lower bound from Proposition 4.6, together with Lemma 4.1, we have that

\[
Y_{BS}(k, c) \geq \frac{k}{-2 \log c} + O\left(\frac{\log(-\log c)}{(-\log c)^{3/2}}\right).
\]

Similarly, since Lemma 4.1 implies that the quantity \( L(k, c) \) from Proposition 4.5 is of asymptotic order

\[
L(k, c) = O(\log c)
\]

as \( c \downarrow 0 \) thanks to Proposition 4.1, the upper bound follows.

The case \( k < 0 \) follows from the put-call symmetry of Proposition 2.1. \( \square \)

Figure 3 illustrates the behaviour of \( Y_{BS}(k, c) \) as \( c \downarrow 0 \), compared with the uniform upper and lower bounds of Proposition 4.6 and the asymptotic formula in Theorem 4.4. We fixed the log-moneyness \( k = 0.2 \) and plotted four functions:

1. \( Y_{upper}(c) = \frac{k}{-\Phi^{-1}\left(\frac{c}{1+c^k}\right)} \) is the upper bound from Proposition 4.6;
2. \( Y_*(c) = Y_{BS}(k, c) \) is the true function of our interest;
3. \( Y_{lower}(c) = \frac{k}{-\Phi^{-1}\left(\frac{c}{1+c^k}\right)} \) is the lower bound from Proposition 4.6;
4. \( Y_{asym}(c) = \frac{k}{\sqrt{-2 \log c}} \) is the asymptotic shape from Theorem 4.4.

Note again that \( Y_{upper} \geq Y_* \geq Y_{lower} \) as predicted. Finally, note that \( Y_{asym} \leq Y_{lower} \) for this range of \( c \).

The next example is due to Gulisashvili [10]. This result is useful in studying the wings of the implied volatility surface for extreme strikes but fixed maturity date.
Figure 3. Bounds and asymptotics of $Y_{BS}(k, \cdot)$ as $c \downarrow 0$.

**Theorem 4.8** (Gulisashvili). If $c(k) \downarrow 0$ as $k \uparrow +\infty$ then

$$Y_{BS}(k, c(k)) = \sqrt{-2\log(e^{-k}c(k))} - \sqrt{-2\log c(k)} + O\left(\frac{\log(-\log c(k))}{\sqrt{-\log c(k)}}\right).$$

If $e^{-k}p(k) \downarrow 0$ as $k \downarrow -\infty$ then

$$Y_{BS}(k, c(k)) = \sqrt{-2\log p(k)} - \sqrt{-2\log(e^{-k}p(k))} + O\left(\frac{\log(-\log(e^{-k}p(k)))}{\sqrt{-\log(e^{-k}p(k))}}\right).$$

where $c(k) = 1 - e^k + p(k)$.

As before, the proof will rely on appropriate uniform bounds:

**Proposition 4.9.** Fix $k \in \mathbb{R}$ and $(1 - e^k)^+ \leq c < 1$. If $k \geq 0$ we have

$$\Phi^{-1}(c) + \sqrt{\left[\Phi^{-1}(c)\right]^2 + 2k} \leq Y_{BS}(k, c) \leq \Phi^{-1}(2c) - \Phi^{-1}(e^{-k}c)$$

and for $k < 0$ we have

$$\Phi^{-1}(e^{-k}p) + \sqrt{\left[\Phi^{-1}(e^{-k}p)\right]^2 - 2k} \leq Y_{BS}(k, c) \leq \Phi^{-1}(2e^{-k}p) - \Phi^{-1}(p)$$

where $p = c + e^k - 1$.

**Proof.** Consider the case $k \geq 0$. For the upper bound, let $d_2 = \Phi^{-1}(e^{-k}c)$ in Theorem 3.1. For the lower bound, let $y = Y_{BS}(k, c)$. Observe that

$$\Phi\left(-\frac{k}{y} + \frac{y}{2}\right) = c + e^k \Phi\left(-\frac{k}{y} - \frac{y}{2}\right) \geq c.$$

The conclusion follows from noting that the strictly increasing map

$$x \mapsto x + \sqrt{x^2 + 2k}$$
from $\mathbb{R}$ to $(0, \infty)$ is the inverse of the map

$$y \mapsto -\frac{k}{y} + \frac{y}{2}$$

from $(0, \infty)$ to $\mathbb{R}$.

The case where $k < 0$ is handled by put-call symmetry as always. \hfill \square

**Remark 4.10.** The idea behind the lower bound is the simple inequality

$$(X - K)^+ \leq X \mathbf{1}_{[K,\infty)}(X)$$

which holds for all $X, K \geq 0$.

**Proof of Theorem 4.8.** For $k \geq 0$, we apply Proposition 4.9 and Lemma 4.1 to get

$$Y_{BS}(k, c(k)) \leq \Phi^{-1}(2c(k)) - \Phi^{-1}(e^{-k}c(k))$$

$$= -\sqrt{-2 \log c(k)} + \sqrt{-2 \log e^{-k}c(k)} + O\left(\frac{\log(-\log(c(k)))}{\sqrt{-\log c(k)}}\right)$$

where we have used $\sqrt{-2 \log(2c)} = \sqrt{-2 \log c + O\left(\frac{1}{\sqrt{-\log c}}\right)}$ as $c \downarrow 0$ to control the error from the first term, and the bound $e^{-k}c(k) \leq c(k)$ to control the error from the second term. Similarly, for the upper bound Proposition 4.9 and Lemma 4.1 yield

$$Y_{BS}(k, c(k)) \geq \Phi^{-1}(c) + \sqrt{\Phi^{-1}(c)^2 + 2k}$$

$$= -\sqrt{-2 \log c(k)} + \sqrt{-2 \log e^{-k}c(k)} + O\left(\frac{\log(-\log(c(k)))}{\sqrt{-\log c(k)}}\right).$$

The $k \downarrow -\infty$ case is similar. \hfill \square

Figure 4 illustrates the behaviour of $Y_{BS}(k, c(k))$ when $c(k) \downarrow 0$ as $k \uparrow \infty$, compared with the uniform upper and lower bounds of Proposition 4.9 and the asymptotic formula in Theorem 4.8. We have chosen the function $c(\cdot)$ according to the variance gamma model. That is, we fix a time horizon $T > 0$ and let

$$c(k) = \mathbb{E}[(X - e^k)^+]$$

where

$$X = e^{\sigma W(G_T) + \Theta G_T + m T}$$

and $\sigma$ and $\Theta$ are real constants, and the process $W$ is a Brownian motion subordinated to the gamma process $G$, which is an independent Lévy process with jump measure $\mu(dx) = \frac{1}{\nu x} e^{-x/\nu} dx$ for some constant $\nu > 0$. The constant $m$ is chosen so that

$$\mathbb{E}[X] = 1.$$

It is well known that $G_T$ has the gamma distribution with mean $T$ and variance $\nu T$. By a routine calculation involving the moment generating functions of the normal and gamma distribution, we find the moment generating function $M$ of $\log X$ to be

$$M(r) = e^{\nu m T}(1 - \nu(\Theta r + \sigma^2 r^2 / 2))^{-T/\nu}.$$ 

Therefore, we must set

$$m = \frac{1}{\nu} \log(1 - \nu(\Theta + \sigma^2 / 2)).$$
Figure 4. Bounds and asymptotics of $Y_{BS}(\cdot, c(\cdot))$ as $c(k) \downarrow 0$ as $k \uparrow \infty$.

Note that we must assume the parameters are such that

$$\Theta + \frac{\sigma^2}{2} < \frac{1}{\nu}$$

to ensure that $m$ is real. Recall that the random variable $X$ has the interpretation of the ratio $X = S_T/F_{0,T}$ of the time-$T$ price $S_T$ of some asset to its initial time-$T$ forward price. The expected value is computed under a fixed time-$T$ forward measure. Hence $c(k)$ models initial normalised price of a call option with log-moneyness $k$ and maturity $T$. We use the parameters $\sigma = 0.1213$, $\nu = 0.1686$ and $\Theta = -0.1436$ as suggested by the calibration of Madan, Carr and Chang [19] and set $T = 1.5$.

As before, we plotted four functions:

1. $Y_{\text{upper}}(k) = \Phi^{-1}(2c(k)) - \Phi^{-1}(e^{-k}c(k))$ is the upper bound from Proposition 4.9;
2. $Y_*(k) = Y_{BS}(k, c(k))$ is the true function of our interest;
3. $Y_{\text{lower}}(k) = \Phi^{-1}(c(k)) + \sqrt{\Phi^{-1}(c(k))^2 + 2k}$ is the lower bound from Proposition 4.9;
4. $Y_{\text{asym}}(k) = \sqrt{-2\log(e^{-k}c(k))} - \sqrt{-2\log c(k)}$ is the asymptotic shape from Theorem 4.8.

As always, note that $Y_{\text{upper}} \geq Y_* \geq Y_{\text{lower}}$ as predicted. Finally, note that $Y_{\text{asym}} \leq Y_{\text{lower}}$ for this example.

The recent paper [7] of De Marco, Hillairet & Jacquier study a similar asymptotic regime as the $k \downarrow -\infty$ case of Theorem 4.8, except now the assumption is that $e^{-k}p(k) \to u > 0$. See also the paper of Gulisashvili [12] for further refinements. The motivation is to study the left-wing behaviour of the implied volatility small in the case where the price of underlying stock may hit zero. The first two terms in the following expansion has been known for a few years; see for instance [25].
**Theorem 4.11** (De Marco, Hillairet & Jacquier). Suppose $e^{-k}p(k) \to u$ as $k \downarrow -\infty$ where $0 < u < 1$. Then letting $c(k) = p(k) + 1 - e^k$ we have

$$Y_{BS}(k, c(k)) = \sqrt{-2k} + \Phi^{-1}(u) + O\left(\frac{1}{\sqrt{-k}} + \varepsilon(k)\right)$$

as $k \downarrow -\infty$, where

$$
\varepsilon(k) = e^{-k}p(k) - u
$$

Our proof of Theorem 4.11 reuses the uniform lower bound from Proposition 4.9. However, another upper bound is needed in this situation:

**Proposition 4.12.** Fix $k \in \mathbb{R}$ and $(1 - e^k)^+ \leq c < 1$. If $k \geq 0$, we have

$$Y_{BS}(k, c) \leq \Phi^{-1}(c + e^k\Phi(-\sqrt{2k})) + \sqrt{2k}$$

and if $k < 0$ we have

$$Y_{BS}(k, c) \leq \Phi^{-1}(e^{-k}p + e^{-k}\Phi(-\sqrt{-2k})) + \sqrt{-2k}$$

where $p = c + e^k - 1$.

**Proof.** In the statement of Theorem 3.1, let $d_2 = -\sqrt{2k}$ if $k \geq 0$, or let $d_1 = \sqrt{-2k}$ if $k < 0$. \hfill \Box

**Proof of Theorem 4.11.** Recall the standard bound on the normal Mills ratio

$$e^{x}\Phi(-\sqrt{2x}) \leq \frac{1}{\sqrt{4\pi x}} \to 0 \text{ as } x \uparrow \infty.$$ 

Hence by Proposition 4.12 we have

$$Y_{BS}(k, c) - \sqrt{-2k} \leq \Phi^{-1}(u + \varepsilon(k) + (-4\pi k)^{-1/2})$$

$$= \Phi^{-1}(u) + O(\varepsilon(k) + (-k)^{-1/2}).$$

Similarly by Proposition 4.8 we have

$$Y_{BS}(k, c(k)) - \sqrt{-2k} \geq \Phi^{-1}(u + \varepsilon(k)) + \sqrt{[\Phi^{-1}(u + \varepsilon(k))]^2 - 2k - \sqrt{-2k}}$$

$$= \Phi^{-1}(u) + O(\varepsilon(k) + (-k)^{-1/2})$$

completing the proof. \hfill \Box

Figure 5 illustrates the behaviour of $Y_{BS}(k, c(k))$ when $e^{-k}p(k) \to u > 0$ as $k \downarrow -\infty$, where $p(k) - c(k) = e^k - 1$, compared with the uniform upper of Proposition 4.12, lower bounds of Proposition 4.9 and the asymptotic formula in Theorem 4.11. We have chosen the function $c(\cdot)$ according to the Black–Scholes model with a jump to default. That is, we fix a horizon $T > 0$ and let

$$c(k) = \mathbb{E}[(X - e^k)^+]$$

where

$$X = 1_{\{T < \tau\}}e^{\sigma W_T + (\lambda - \sigma^2/2)T}$$

and $\sigma$ and $\lambda$ are positive constants, the process $W$ is a Brownian motion and the random variable $\tau$ is independent of $W$ and exponentially distributed with rate $\lambda$, so that

$$\mathbb{E}[X] = 1.$$
Note that
\[ e^{-k}p(k) = \mathbb{E}[(1 - e^{-k}X)^+] \]
\[ \rightarrow \mathbb{P}(X = 0) \]
\[ = \mathbb{P}(\tau \leq T) \]
\[ = 1 - e^{-\lambda T}. \]

On the other hand, it is straightforward to calculate
\[ c(k) = C_{BS}(k - \lambda T, \sigma \sqrt{T}). \]

We use the parameters \( \sigma = 0.20 \) and \( \lambda = 0.05 \) with time horizon \( T = 1.5. \)

As before, we plotted four functions:

1. \( Y_{upper}(k) = \Phi^{-1}[e^{-k}p(k) + e^{-k}\Phi(-\sqrt{-2k})] + \sqrt{-2k} \) the upper bound from Proposition 4.12;
2. \( Y_*(k) = Y_{BS}(k, c(k)) \) is the true function of our interest;
3. \( Y_{lower}(k) = \Phi^{-1}(e^{-k}p(k)) + \sqrt{\Phi^{-1}(e^{-k}p(k))^2 - 2k} \) is the lower bound from Proposition 4.9;
4. \( Y_{asym}(c) = \sqrt{-2k} + \Phi^{-1}(u) \) is the asymptotic shape from Theorem 4.11.

As always, note that \( Y_{upper} \geq Y_* \geq Y_{lower} \) as predicted, that \( Y_{upper} \) is a surprisingly good approximation for \( Y_* \), and that \( Y_{asym} \leq Y_{lower} \) for this example. Indeed, \( Y_{asym} \) does not seem to be a very good approximation of \( Y_* \) for this range of \( k \).

We conclude with some remarks on the bounds and asymptotic formulae in this section. The numerical results suggest that for at least some situations, one of the upper or lower bounds is a better approximation to the implied total standard deviation than the corresponding lowest order asymptotic formula. One could argue that with more terms in the asymptotic series, better accuracy could be attained with the asymptotics. Although such a claim is indeed plausible, there are a few reasons why it is beside the point.
First, the numerical results presented here should only be considered a proof of concept, rather than a head-to-head competition between state-of-the-art approximations. Nevertheless, it is worth noting both the given bounds and the asymptotic formulae are only approximations, and therefore have an error term. But unlike the error terms of an asymptotic formula, the error term for our bounds have a known sign.

Second, given one bound, the theorems of Section 3 give a systematic way of finding a better bound. Indeed, fix \((k, c)\) with \(k > 0\), and let \(y_\ast = Y_{BS}(k, c)\). Suppose it known that \(y_\ast < y_1\) where \(y_1\) is some given approximation. Define \(F : (y_{\text{min}}, \infty) \to (0, \infty)\) by

\[
F(y) = H_1(-k/y + y/2; k, c)
\]

where

\[
y_{\text{min}} = \Phi^{-1}(c) + \sqrt{[\Phi^{-1}(c)]^2 + 2k}
\]

and \(H_1\) is the functions defined by equation (2) of Section 3. Letting \(y_2 = F(y_1)\) we have by Theorem 3.1 that

\[
y_\ast < y_2.
\]

However, more is true. Note that the map \(F\) has a unique fixed point \(y_\ast\). Since

\[
\lim_{y \downarrow y_{\text{min}}} F(y) = \infty
\]

we conclude by the continuity of \(F\) that \(F(y) > y\) for \(y_{\text{min}} < y < y_\ast\), and more importantly, that \(F(y) < y\) for \(y > y_\ast\). In particular,

\[
y_2 = F(y_1) < y_1.
\]

That is, \(y_2\) is a better approximation of \(y_\ast\) and the error term has the same sign as the original approximation. Of course, this process can iterated. Letting \(y_n = F(y_{n-1})\) we see that the sequence \((y_n)_{n \geq 1}\) is decreasing and \(\inf_n y_n = y_\ast\).

Notice that this sequence converges very rapidly. Indeed, by Taylor’s theorem

\[
y_n = F(y_{n-1})
\]

\[
= F(y_\ast) + F'(y_\ast)(y_{n-1} - y_\ast) + \frac{1}{2}F''(\hat{y})(y_{n-1} - y_\ast)^2
\]

for some \(y_\ast < \hat{y} < y_{n-1}\). Since \(y_\ast\) minimises \(F\) we have

\[
F'(y_\ast) = 0
\]

and hence, by the continuity of \(F''\), we have

\[
\frac{y_n - y_\ast}{(y_{n-1} - y_\ast)^2} \to \frac{1}{2}F''(y_\ast) = \frac{1}{2y_\ast} \left(\frac{k}{y_\ast} + \frac{y_\ast}{2}\right)^2
\]

as \(n \to \infty\).

Furthermore, we can find our initial upper bound \(y_1\) by choosing any \(y_0 > y_{\text{min}}\) and letting \(y_1 = F(y_0)\). This procedure is illustrated by the cobweb diagram of Figure 6. Of course, the convergence can be helped along by an inspired choice of \(y_0\) as discussed at the beginning of this section.
Figure 6. The cobweb diagram illustrating the convergence of \( y_0, y_1, y_2, \ldots \) to the fixed point \( y^* = F(y^*) \).

The above discussion of a rapidly converging sequence should be contrasted with the approach taken, for instance, in the paper of Gao & Lee [9]. There a systematic method of computing terms in the asymptotic series for implied volatility is obtained. However, unlike the procedure discussed above, an asymptotic series may diverge as more terms are added.

A third and final point is that the approximations for the implied total standard deviation are not particularly interesting on their own. Indeed, to use the formulae in Proposition 4.9 one must already know the normalised bond price \( c(k) \). If this quantity is to be calculated numerically from a certain model, one might as well as well compute the \( Y_{BS}(k, c(k)) \) numerically also. The point of these bounds is to be used in conjunction with other, model dependent bounds on \( c(k) \) to obtain useful bounds on the quantities of interest.

5. A CHARACTERISATION OF LOG-CONCAVE DISTRIBUTIONS

This section is only tangentially related to the main thread of this note, but it is included here because of its possible independent interest. In particular, we derive a characterisation of log-concave probability distributions on \( \mathbb{R} \).

To motivate the discussions, note that Theorem 3.1 has an equivalent formulation:

**Theorem 5.1.** For all \( k \in \mathbb{R} \) and \( y \geq 0 \) we have

\[
C_{BS}(k, y) = \sup_{p \in [0,1]} \left[ \Phi\left( \Phi^{-1}(p) + y \right) - e^{kp} \right].
\]

**Proof.** Fix \( k \) and \( y \) and let \( c = C_{BS}(k, y) \). Note that \( y = Y_{BS}(k, c) \) and by Theorem 3.1 we have

\[
y \leq \Phi^{-1}(c + e^k \Phi(d_2)) - d_2.
\]

with equality if and only if \( d_2 = -k/y - y/2 \). Letting \( \Phi(d_2) = p \), the result follows upon rearrangement. \( \square \)
The function \( p \mapsto \Phi(\Phi^{-1}(p) + y) \) appeared as the value function of the problem of maximising the probability of a perfect hedge considered by Kulldorff \[15\]. (Also see Section 2.6 of the book of Karatzas \[14\].) This function is increasing and concave, and as in the Lagrangian duality approach to utility maximisation, it is natural to compute its convex dual function. Theorem 5.1 says that this dual function is essentially (that is, in the variable \( K = e^k \)) the Black–Scholes call price function.

The next result is a natural generalisation of this observation.

**Theorem 5.2.** Let \( F \) be the distribution function of a probability measure with positive, continuous density \( f \). The function

\[
p \mapsto F(F^{-1}(p) + y)
\]

is concave on \([0, 1]\) for all \( y \geq 0 \) if and only if \( f \) is log-concave.

**Proof.** Fix \( y \geq 0 \) and let

\[
G(p) = F(F^{-1}(p) + y)
\]

Note that the derivative is given by the formula

\[
G'(p) = \frac{f(F^{-1}(p) + y)}{f(F^{-1}(p))}.
\]

Therefore, the function \( G \) is concave if and only if \( G' \) is decreasing, or equivalently,

\[
\log f(b + y) - \log f(b) \leq \log f(a + y) - \log f(a)
\]

for all \( b \geq a \). This last condition holds for all \( y \geq 0 \) if and only if \( \log f \) is concave. \( \square \)

Let \( f \) be a positive log-concave density satisfying the Inada-like condition

\[
\lim_{x \to -\infty} \frac{f(x + y)}{f(x)} = +\infty \quad \text{and} \quad \lim_{x \to +\infty} \frac{f(x + y)}{f(x)} = 0
\]

for all \( y > 0 \). Inspired by Theorem 5.1, given a log-concave density \( f \) we can define a function \( C_f : \mathbb{R} \times [0, \infty) \to [0, 1) \) by

\[
C_f(k, y) = \sup_{p \in [0, 1]} F(F^{-1}(p) + y) - pe^k.
\]

Of course we have \( C_\phi = C_{BS} \) where \( \phi \) is the standard normal density.

**Theorem 5.3.** The exists a non-negative martingale \((M_y)_{y \geq 0}\) defined on some filtered probability space such that

\[
C_f(k, y) = \mathbb{E}[(M_y - e^k)^+].
\]

**Proof.** Notice that

- the map \( y \mapsto C_f(k, y) \) is increasing with \( C_f(k, 0) = (1 - e^k)^+ \) for all \( k \in \mathbb{R} \), and
- the map \( K \mapsto C_f'(\log K, y) \) is decreasing and convex for all \( y \geq 0 \).

Note the Inada-like conditions guarantee that \( C_f \) is everywhere finite. The conclusion follows from a theorem of Kellerer \[13\]. \( \square \)

In the case when \( f = \phi \), a suitable martingale \( M \) is a time-changed geometric Brownian motion. That is, letting \( S_t = e^{W_t - t/2} \) be the Black–Scholes model, where \( W \) is a Brownian motion, we can set \( M_y = S_{y^2} \). 21
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