A note on invariant measures for HJM models[1]

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Abstract. This note analyzes the mean-reverting behavior of time-homogeneous Heath-Jarrow-Morton (HJM) forward rate models in the weighted Sobolev spaces $\{H_w\}_w$. An explicit sufficient condition is given under which invariant measures exist for the HJM dynamics. In particular, every HJM model with constant volatility and market price of risk has a family of invariant measures parametrized by the distribution of the long rate.

Key words: term structure of interest rates, stochastic partial differential equations, invariant measures

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1 Introduction

It is commonly believed that over a long enough time horizon interest rates tend to revert to their average historical levels. In this note we examine the long time behavior of time-homogeneous Heath-Jarrow-Morton (HJM) interest rate models and quantify this notion of mean-reversion. In particular, we find sufficient conditions on the model parameters for the existence of and weak convergence to invariant measures.

The forward rate $f_t(x)$ at time $t \geq 0$ for time to maturity $x \geq 0$ is given by

$$f_t(x) = -\frac{\partial}{\partial x} \log\left(P(t, t+x)\right)$$

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where $P(t,T)$ denotes the price at time $t \geq 0$ of a zero-coupon bond with maturity date $T = t+x$. In the HJM framework [9] with the Musiela parametrization [12], the forward rates satisfy, in a sense to be made precise below, the following stochastic partial differential equation

$$df_t(x) = \left(\frac{\partial}{\partial x} f_t(x) + a_t(x)\right) dt + \sum_{i=1}^{\infty} \sigma_t^i(x) dW_t^i \tag{1}$$

where the drift is given by the the famous HJM no-arbitrage condition

$$a_t(x) = \sum_{i=1}^{\infty} \sigma_t^i(x) \left(\int_0^x \sigma_t^i(u) du + \lambda_t^i\right).$$

For several special cases, invariant measures have been proven to exist for the HJM dynamics. For instance, Musiela in [12] and Vargiolu in [13] studied the linear Gaussian HJM model and found conditions under which there exist invariant measures for the risk-neutral dynamics. Brace, Gatarek, and Musiela in [2] proposed a non-linear HJM model, the so-called market model, and found conditions under which there exist an invariant measure for the risk-neutral dynamics.

We proceed as follows: In section 2 we recall the HJM model in the Musiela parametrization and introduce the family of state space $\{H_w\}_w$. In section 3 we present the main theorem: under an explicit sufficient condition there exists a family of invariant measures for the HJM dynamics. Since the conditions of the theorem are trivially satisfied when the volatility and market price of risk are constant, we conclude this section with a description of the invariant measures of the linear HJM models on this class of state spaces.

2 The HJM model

In this section we recall the mild formulation of the HJM equation as a stochastic evolution equation

$$df_t = (\frac{\partial}{\partial x} f_t + a_t)\ dt + \sigma_t dW_t$$

in a Hilbert space H, where $\frac{\partial}{\partial x}$ generates a strongly continuous semigroup on H. Such a formulation places us within the framework of Da Prato and Zabczyk [5, 6].

2.1 The choice of state space

The state space H needs to be chosen with care. For instance, the bond price at time $t \geq 0$ for maturity $T \geq t$ is given by

$$P(t,T) = \exp\left(-\int_0^{T-t} f_t(s)\ ds\right),$$

and thus H should be a space of locally-integrable functions. Additionally, it is convenient to choose a space H of continuous functions so that point-wise evaluation is well-defined, and in particular, so that the definition of the spot interest rate $r_t = f_t(0)$ is meaningful.

Filipovic [8] proposed a family of spaces $\{H_w\}_w$ as appropriate state spaces to analyze the HJM dyanamics. These spaces are defined as follows:

Definition 1 *For a positive increasing function* $w : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\int_0^\infty \frac{dx}{w(x)} < +\infty$, *let* H_w *denote the space of absolutely continuous functions* $f : \mathbb{R}_+ \to \mathbb{R}$ *satisfying*

$$\int_0^\infty f'(x)^2 w(x)dx < +\infty$$

where f' *is the weak derivative of* f. *Endow the space* H_w *with the inner product*

$$\langle f, g\rangle_{H_w} = f(\infty)g(\infty) + \int_0^\infty f'(x)g'(x)w(x)dx.$$

Remark 1 *For every* $f \in H_w$, *the limit* $f(\infty) = \lim_{x\to\infty} f(x)$ *is well-defined. Indeed, the the improper integral* $f(\infty) = f(0)+\int_0^\infty f'(x)dx$ *converges absolutely since*

$$\int_0^\infty |f'(x)|dx \le \left(\int_0^\infty f'(x)^2 w(x)dx\right)^{1/2} \left(\int_0^\infty \frac{dx}{w(x)}\right)^{1/2}.$$

In [8] the space H_w *was endowed with the inner product*

$$(f, g)_{H_w} = f(0)g(0) + \int_0^\infty f'(x)g'(x)w(x)dx.$$

Since the inner products $(\cdot,\cdot)_{H_w}$ *and* $\langle\cdot,\cdot\rangle_{H_w}$ *give rise to equivalent norms on* H_w, *we choose to work with the inner product* $\langle\cdot,\cdot\rangle_{H_w}$ *to neaten the presentation of the result.*

The spaces $\{H_w\}_w$ have many nice properties as seen by the following proposition:

Proposition 2 *Fix a weight function* w. *The inner product space* H_w *is a separable Hilbert space. In addition, the evaluation functional* δ_x *and the definite integration functional* I_x *defined by*

$$\delta_x(f) = f(x) \text{ and } I_x(f) = \int_0^x f(s)ds$$

are continuous on H_w *for all* $x \ge 0$. *Furthermore, the semigroup of operators on* H_w *defined by*

$$(S_t f)(x) = f(t + x)$$

is strongly continuous.

The proofs of the above statements can be found in [8].

The most important motivation for the choice of H_w as state space is that H_w is compatible with the HJM no-arbitrage condition, at least if w increases quickly enough. To state the precise result, we introduce some more notation.

For every w, we distinguish the subspace $H_w^0 \subset H_w$ given by

$$H_w^0 = \{f \in H_w \text{ such that } f(\infty) = 0\}.$$

The subspace H_w^0 is the closed subspace of H_w orthogonal to the functional $\delta_\infty \in H_w^*$. Note that we have the natural decomposition $H_w \cong H_w^0 \oplus \mathbb{R}$.

Fix an arbitrary real separable Hilbert space G with inner product $\langle \cdot, \cdot \rangle_G : G \times G \to \mathbb{R}$, and let $\mathcal{L}^2(G, H_w^0)$ denote the space of Hilbert-Schmidt operators from G into H_w^0 with with norm given by

$$\|A\|_{\mathcal{L}^2(G,H_w^0)} = \left(\sum_{i=1}^{\infty} \|Ag^i\|_{H_w}^2 \right)^{1/2}$$

where $\{g^i\}_{i\in\mathbb{N}}$ is any orthonormal basis for G.

Define the HJM function F_{HJM} as a map from $\mathcal{L}(G, H_w^0)$ into the space of real-valued functions on $\mathbb{R}_+$ via

$$F_{\text{HJM}}(A)(x) = \langle A^*\delta_x, A^* I_x \rangle_G$$

where for each $L \in H_w^*$ we let A^*L be the unique element of G such that $\langle A^*L, g\rangle_G = L(Ag)$ for every $g \in G$.

When the weight w satifies a mild growth condition, the function F_{HJM} is a locally Lipschitz map from $\mathcal{L}^2(G, H_w^0)$ into H_w^0. The following proposition, which is proven in [8], provides an estimate on the smoothness of F_{HJM}.

Proposition 3 *Fix a weight w such that $\int_0^\infty \frac{dx}{w(x)^{1/3}} < +\infty$. For $A, B \in \mathcal{L}^2(G, H_w^0)$ we have*

$$\|F_{HJM}(A) - F_{HJM}(B)\|_{H_w} \leq C_w \|A - B\|_{\mathcal{L}^2(G,H_w)} \|A + B\|_{\mathcal{L}^2(G,H_w)}$$

where $C_w = 2\left(\int_0^\infty \frac{dx}{w(x)^{1/3}} \right)^{3/2}$.

2.2 The specification of the model

We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and such that there exists a Wiener process W defined cylindrically on the separable Hilbert space G. Let $\mathcal{P}$ be the predictable σ-field on $\mathbb{R}_+ \times \Omega$.

We now state a definition, which we essentially take from [8], of an HJM model for the forward rate:

Definition 4 *An HJM model on H_w is a pair of functions (λ, σ) where*

i. the measurable function λ from $(\mathbb{R}_+ \times \Omega \times H_w, \mathcal{P} \otimes \mathcal{B}_{H_w})$ into $(G, \mathcal{B}_G)$, *ii. and the measurable function σ from $(\mathbb{R}_+ \times \Omega \times H_w, \mathcal{P} \otimes \mathcal{B}_{H_w})$ into $(\mathcal{L}^2(G, H_w^0), \mathcal{B}_{\mathcal{L}^2(G,H_w^0)})$,*

such that there exists a non-empty set of initial conditions $f_0 \in H_w$ for which there exists a unique, continuous H_w-valued solution $\{f_t\}_{t\geq 0}$ of the HJM equation:

$$f_t = S_t f_0 + \int_0^t S_{t-s} a(s, \omega, f_s) ds + \int_0^t S_{t-s} \sigma(s, \omega, f_s) dW_s. \tag{2}$$

where

$$a(t, \omega, f) = F_{\scriptscriptstyle HJM} \circ \sigma(t, \omega, f) + \sigma(t, \omega, f)\lambda(t, \omega, f).$$

Several remarks are in order.

Remark 2 *For all HJM models on H_w, the long rate is constant. Indeed, by the continuity of the functional δ_∞, we have by equation (2) that $f_t(\infty) = f_0(\infty)$ a.s. for all $t \geq 0$. Consequently, the HJM models on H_w verify the result of Dybvig, Ingersoll, and Ross [7] that the long rate can never fall. See Hubalek, Klein, and Teichmann [10] for a proof of this general theorem.*

Remark 3 *The choice of the auxiliary space G does not play a crucial role here. Indeed, the stochastic integral with respect to the cylindrical Wiener process W can be decomposed as a sum of stochastic integrals with respect to independent scalar Wiener processes $\{W^i\}_{i\in\mathbb{N}}$ by fixing an orthonormal basis $\{g^i\}_{i\in\mathbb{N}}$ of G and letting $\sigma_t^i = \sigma(t, \omega, f_t)g^i$ and (formally) $W_t^i = \langle g^i, W_t\rangle_G$. We can recover the classical n-factor HJM model in this setting by setting $\sigma^i \equiv 0$ for $i > n$; equivalently, we may take G to be $\mathbb{R}^n$. We prefer to leave G unspecified so that the reader may substitute a Hilbert space to suit her intuition. For instance, one may choose $G = L^2(\mathbb{R}_+)$ and interpret the increment $dW_t(x)$ as a white-noise in space and time. The spacial variable x may be viewed as the time to maturity, and the volatility $\sigma(t, \omega, f_t)$ as an integral operator relating the instantaneous correlations between different maturities.*

Remark 4 *The mild formulation (2) of the HJM equation proves more convenient than the classical formulation (1) since the operator $\frac{\partial}{\partial x}$ is unbounded on H_w.*

A consequence of Proposition 3 is that sufficient conditions for the existence of HJM models in the space H_w are easy to check as seen in the following proposition:

Proposition 5 *Fix a weight w such that $\int_0^\infty \frac{dx}{w(x)^{1/3}} < +\infty$, and assume*

$$\|\sigma(t, \omega, f) - \sigma(t, \omega, g)\|_{\mathcal{L}^2(G,H_w^0)} \leq L_\sigma \|f - g\|_{H_w} \tag{3}$$

$$\|\sigma(t, \omega, f)\|_{\mathcal{L}^2(G,H_w^0)} \leq M_\sigma \tag{4}$$

$$\|\sigma(t, \omega, f)\lambda(t, \omega, f) - \sigma(t, \omega, g)\lambda(t, \omega, g)\|_{H_w} \leq L_{\sigma\lambda} \|f - g\|_{H_w} \tag{5}$$

for some positive constants L_σ, M_σ, and $L_{\sigma\lambda}$ and all $t \geq 0$, $\omega \in \Omega$ and $f, g \in H_w$. Then the pair (λ, σ) is an HJM model on H_w. Furthermore, for any initial forward curve $f_0 \in H_w$ there exists a unique, continuous solution to the equation (2) such that $\mathbb{E}\{\sup_{t\in[0,T]} \|f_t\|_{H_w}^p\} < +\infty$ for all finite $T \geq 0$ and $p \geq 0$.

The proposition can be proved by a standard fixed-point argument which can be found, for instance, in Da Prato and Zabczyk [5].

3 Invariant measures for HJM models

In this section, we note that the spaces H_w are also compatible with a notion of mean-reversion if the weight w grows sufficiently quickly.

We restrict our attention to an important class of HJM models:

Definition 6 *A time-homogeneous HJM model* (λ, σ) *is an HJM model such that there exist measurable functions* $\bar{\lambda}$ *and* $\bar{\sigma}$ *such that* $\lambda(t, \omega, f) = \bar{\lambda}(f)$ *and* $\sigma(t, \omega, f) = \bar{\sigma}(f)$ *for all* $(t, \omega, f) \in \mathbb{R}_+ \times \Omega \times H_w$.

We henceforth deal exclusively with time-homogeneous HJM models. To lighten the notation, we write $\bar{\lambda} = \lambda$ and $\bar{\sigma} = \sigma$ and $a = F_{\text{HJM}} \circ \sigma + \sigma\lambda$. Associated with such an HJM model is the equation

$$f_t = S_t f_0 + \int_0^t S_{t-s} a(f_s) ds + \int_0^t S_{t-s} \sigma(f_s) dW_s. \tag{6}$$

Consider equation (6) with the initial condition given by an H_w-valued $\mathcal{F}_0$-measurable random variable f_0 with law μ. If for all $t \geq 0$ the law of the random variable f_t is also μ then we say μ is an invariant measure for the HJM model.

The following result finds explicit sufficient conditions such that an HJM model has an invariant measure.

Theorem 7 *Fix the weight* w *and let* $C_w = 2\left(\int_0^\infty \frac{dx}{w(x)^{1/3}}\right)^{3/2}$ *and* $\alpha_w = \inf_{x\geq 0} \frac{w'(x)}{w(x)}$. *Let* (λ, σ) *be a time-homogeneous HJM model on* H_w *with constants* L_σ, M_σ, *and* $L_{\sigma\lambda}$ *be given by equations (3), (4), (5). If we have*

$$L_\sigma^2 + 4C_w L_\sigma M_\sigma + 2L_{\sigma\lambda} < \alpha_w \tag{7}$$

then there exists an infinite family of invariant measures $\{\mu^\nu\}_\nu$ *on* H_w *for the HJM model* (λ, σ) *with the following properties:*

i. Let ν *be a probability measure on* $(\mathbb{R}, \mathcal{B}_\mathbb{R})$. *For every* $\mathcal{F}_0$*-measurable initial forward curve* $f_0 \in H_w$ *such that the marginal distribution of the inital long rate* $f_0(\infty)$ *is* ν, *there exists a unique measure* μ^ν *such that the law of* f_t *converges weakly to* μ^ν.

ii. For every bounded $\phi : H_w \to \mathbb{R}$ *such that* $|\phi(f) - \phi(g)| \leq \|f - g\|_{H_w}$ *for all* $f, g \in H_w$ *we have*

$$\left|\mathbb{E}\{\phi(f_t)\} - \int_{H_w} \phi(f)\mu^\nu(df)\right| \leq (1 + \mathbb{E}\{\|f_0\|_{H_w}\})e^{-\alpha_w t/2}.$$

Proof. The semigroup $\{S_t\}_{t\geq 0}$ is a strict contraction on H_w^0. Indeed, we have the calculation

$$\|S_t h\|_{H_w^0}^2 = \int_0^\infty h'(x+t)^2 w(x)dx \leq \sup_{x\geq 0} \frac{w(x)}{w(x+t)} \|h\|_{H_w^0}^2$$

for every $h \in H_w^0$, and hence $\|S_t\|_{\mathcal{L}(H_w^0)} \leq e^{-\alpha_w t/2}$.

For every $c \in \mathbb{R}$, let $\sigma^c(h) = \sigma(h + c\mathbf{1})$ and $a^c(h) = a(h + c\mathbf{1})$ for $h \in H_w^0$, and where $\mathbf{1}(x) = 1$ for all $x \geq 0$. We have

$$\langle A_n h, h\rangle_{H_w} \leq -\frac{\alpha_w}{2 + \alpha_w/n}\|h\|_{H_w}^2 \text{ for all } h \in H_w^0$$

where $A_n = n^2 \int_0^\infty e^{-nt}(S_t - I)dt$ is the n^{th} Yosida approximation of the generator of $\{S_t\}_{t\geq 0}$. By Proposition 3 and assumption (7) there is an $\epsilon > 0$ such that

$$\begin{aligned} &2\langle A_n(g-h) + a^c(g) - a^c(h), g - h\rangle_{H_w} \\ &\quad + \|\sigma^c(g) - \sigma^c(h)\|_{\mathcal{L}^2(G,H_w)}^2 \leq -\epsilon \|g-h\|_{H_w}^2 \end{aligned}$$

for all $g, h \in H_w^0$.

Let $f_0 \in H_w$ be deterministic with $c = f_0(\infty)$. Applying the decomposition $H_w \cong H_w^0 \oplus \mathbb{R}$ to equation (6) yields the system of equations $f_t(\infty) = c$ and

$$f_t^0 = S_t f_0^0 + \int_0^t S_{t-s} a^c(f_s^0)ds + \int_0^t S_{t-s}\sigma^c(f_s^0)dW_s \tag{8}$$

where $f_t^0 = \mathrm{Proj}_{H_w^0}(f_t)$. By Theorem 6.3.2 of [6], there exists for every $c \in \mathbb{R}$ a unique invariant measure $\mu^{0,c}$, supported on H_w^0, for equation (8). The measure $\mu^c = \mu^{0,c} * \delta_{c\mathbf{1}}$ is an invariant measure for the HJM model. By the Feller property, we have that $c \mapsto \mathbb{E}\{\phi(f_t)\}$ is continuous for every $t \geq 0$ and every bounded, continuous $\phi : H_w \to \mathbb{R}$. Since as $t \to \infty$ we have $\mathbb{E}\{\phi(f_t)\} \to \mu^c(\phi)$, the map $c \mapsto \mu^c(\phi)$ is measurable.

Now let the initial condition for equation (6) be the $\mathcal{F}_0$-measurable random variable $f_0 = f_0^0 + C\mathbf{1}$ where C is a real random variable with law ν. Let $\phi : H_w \to \mathbb{R}$ be bounded and such that $|\phi(f) - \phi(g)| \leq \|f - g\|_{H_w}$ for all $f, g \in H_w$. Again by Theorem 6.3.2 of [6], we have

$$\left|\mathbb{E}\{\phi(f_t)|C\} - \mu^C(\phi)\right| \quad \leq \quad (1 + \mathbb{E}\{\|f_0\|_{H_w}|C\})e^{-\alpha_w t/2}.$$

almost surely. Taking the expectation of both sides and letting the measure μ^ν be defined by $\mu^\nu(\phi) = \mathbb{E}\{\mu^C(\phi)\}$ completes the proof. ■

Remark 5 *In a practical sense, the HJM model admits a unique invariant measure. Indeed, it is meaningless to consider initial forward curves with differing long rates since, within the context of a given HJM model, the long rate is constant. In other words, the value $c = f_0(\infty)$ can be considered a model parameter, elevated to the status of the functions λ and σ. Given the three parameters λ, σ, and c, the unique invariant measure for the HJM model is the measure μ^ν from the theorem, where ν is the point mass concentrated at c.*

Note in the special case where $\lambda(f) = \lambda$ and $\sigma(f) = \sigma$ are constant functions, assumption (7) is trivially satisfied since $L_\sigma = L_{\sigma\lambda} = 0$.

Corollary 8 *Let the weight w be such that $\inf_{x\geq 0} \frac{w'(x)}{w(x)} > 0$, and let $\lambda \in G$ and $\sigma \in \mathcal{L}^2(G, H_w^0)$ be deterministic constants.*

The equation

$$f_t = S_t f_0 + \int_0^t S_{t-s}(F_{\scriptscriptstyle HJM}(\sigma) + \sigma\lambda) ds + \int_0^t S_{t-s}\sigma dW_s \tag{9}$$

has a family of invariant measures $\{\mu^\nu\}_\nu$ on the space H_w. For fixed ν, the random variables $f(x) = \delta_x(f)$ on $(H_w, \mathcal{B}_{H_w}, \mu^\nu)$ have the following properties:

i. The distribution of the long rate $f(\infty)$ is ν.

ii. The conditional distribution of the forward rate $f(x)$ given the long rate is Gaussian, for every $x \geq 0$.

iii. If $\int_{-\infty}^{\infty} |c| \; \nu(dc) < +\infty$ and if $\lambda = 0$ then the expected value of the short rate $f(0)$ is greater than or equal to the expected value of any other forward rate.

iv. If $\int_{-\infty}^{\infty} c^2 \; \nu(dc) < +\infty$ then the variance of the short rate is greater than or equal to the variance of any other forward rate.

Proof. Since σ and λ are constant functions by Theorem 7 equation (9) has a family of invariant measures of the form $\mu^\nu = \mu^0 * \nu$. The measure μ^0 is the unique invariant measure on H_w^0 for the Ornstein-Uhlenbeck process $\{f_t^0\}_{t\geq 0}$ and is Gaussian by Theorem 9.3.1 of Da Prato and Zabczyk [6].

By the continuity of the evaluation functionals δ_x we have

$$\begin{aligned}
\mathbb{E}\{f_t(x)\} - \mathbb{E}\{f_0(t+x)\} &= \delta_x \int_0^t S_{t-s}(F_{\rm HJM}(\sigma) + \sigma\lambda) ds \\
&= \int_0^t \delta_{t-s+x}(F_{\rm HJM}(\sigma) + \sigma\lambda) ds \\
&= \int_x^{t+x} \langle \sigma^*\delta_s, \sigma^* I_s + \lambda \rangle_G ds \\
&= \frac{1}{2}\left(\|\sigma^* I_{x+t} + \lambda\|_G^2 - \|\sigma^* I_x + \lambda\|_G^2 \right).
\end{aligned}$$

Noting that the functional δ_{t+x} converges strongly in H_w^* to δ_∞ and the functional I_{t+x} converges strongly in H_w^{0*} to a functional $I_\infty \in H_w^{0*}$ we have

$$\begin{aligned}
\mathbb{E}_{\mu^\nu}\{f(x)\} &= \lim_{t\to\infty} \mathbb{E}\{f_t(x)\} \\
&= \mathbb{E}_{\mu^\nu}\{f(\infty)\} + \frac{1}{2}\left(\|\sigma^* I_\infty + \lambda\|_G^2 - \|\sigma^* I_x + \lambda\|_G^2 \right)
\end{aligned}$$

In the case that $\lambda = 0$, it follows that

$$\begin{aligned}
\mathbb{E}_{\mu^\nu}\{f(x)\} &= \mathbb{E}_{\mu^\nu}\{f(\infty)\} + \frac{1}{2}\left(\|\sigma^* I_\infty\|_G^2 - \|\sigma^* I_x\|_G^2 \right) \\
&\leq \mathbb{E}_{\mu^\nu}\{f(\infty)\} + \frac{1}{2}\|\sigma^* I_\infty\|_G^2 = \mathbb{E}_{\mu^\nu}\{f(0)\}
\end{aligned}$$

Finally, we have the variance of $f(x)$ is given by

$$\mathrm{Var}_{\mu^\nu}\{f(x)\} = \mathrm{Var}_{\mu^\nu}\{f(\infty)\} + \int_x^\infty \|\sigma^* \delta_t\|_G^2 dt$$

which is decreasing in $x \geq 0$. ■

Remark 6 *We note that setting $\lambda = 0$ in Corollary 8 is equivalent to considering the risk-neutral HJM dynamics given by*

$$f_t = S_t f_0 + \int_0^t S_{t-s} F_{HJM}(\sigma)\ ds + \int_0^t S_{t-s}\sigma\ d\tilde{W}_s \tag{10}$$

where the process $\tilde{W}_t = W_t + \int_0^t \lambda_s ds$ is a cylindrical G-valued Wiener process under the risk-neutral measure $\mathbb{Q}$ with Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^t \|\lambda_s\|_G^2\ ds + \int_0^t \langle \lambda_s, d\tilde{W}_s\rangle_G\right).$$

Remark 7 *Vargiolu [13] studied the risk-neutral dynamics of the linear Gaussian HJM model when the state space is one of the family of Sobelev spaces $\{H_\gamma^1\}_{\gamma\geq 0}$ with norm*

$$\|f\|_{H_\gamma^1}^2 = \int_0^\infty f(x)^2 e^{-\gamma x} dx + \int_0^\infty f'(x)^2 e^{-\gamma x} dx.$$

and found, for the case $\gamma > 0$, that if

$$\sum_{i=1}^\infty \|\sigma^i\|_{H_\gamma^1}^2 < +\infty,\ \sum_{i=1}^\infty \|\sigma^i\|_{H_0^1}^4 < +\infty,\ \text{and} \sum_{i=1}^\infty \|\sigma^i\|_{L_\gamma^4}^4 < +\infty$$

then there exists a mild solution to equation (10) in the space H_γ^1. If in addition $\sum_{i=1}^\infty \|\sigma^i\|_{H_0^1}^2 < +\infty$ then there exists a family of Gaussian invariant measures on H_γ^1.

We note that by choosing the state space to be one of the (much smaller) Sobolev spaces H_w with a weight w that grows fast enough, then there exists a unique mild solution of equation (8) if $\sigma \in \mathcal{L}^2(G, H_w^0)$ and $\lambda \in G$. Furthermore, under no additional assumptions, there exists a family of invariant measures on H_w.

4 Conclusion

We studied the mild solution to equation (6) in the family of Hilbert spaces $\{H_w\}_w$. These spaces are attractive candidates for the state space for the HJM dynamics since the HJM function F is locally Lipschitz on $\mathcal{L}^2(G, H_w^0)$ when the weight w grows fast enough. We find that when the weight grows exponentially fast, the shift semigroup $\{S_t\}_{t\geq 0}$ has a strongly dissipative generator on H_w^0, and an explicit sufficient condition for the existence of invariant measures for the HJM dynamics can be formulated in terms of the Lipschitz constants and bounds for the volatility and market price of risk. In particular, this condition is satisfied for all linear Gaussian HJM models on these spaces.

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