THE DISTRIBUTION OF EXPONENTIAL LÉVY FUNCTIONALS

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ABSTRACT. The distribution of random variables of the form $\int_0^\infty e^{\xi_{s-}} d\eta_s$ is considered for a Lévy process (ξ, η). In particular, we give a simple proof of a result of Carmona, Petit, and Yor [3] that the distribution function verifies an integro-differential equation related to the characteristic triple of (ξ, η).

Let W be a standard Wiener process and c a positive constant. The purpose of this note is to offer a short, self-contained proof of the following result:

Theorem 1 (Dufresne [5]). *The random variable $\int_0^\infty e^{W_s - cs} ds$ has a density f given by*

$$f(x) = \frac{2^{2c}}{\Gamma(2c)} \frac{e^{-2/x}}{x^{1+2c}}, \quad \textit{for } x > 0.$$

More generally, let $(\xi^1, \ldots, \xi^n, \eta^1, \ldots, \eta^n)$ be a $2n$-dimensional Lévy processes. We aim to describe the joint distribution function

$$F(t, x) = \mathbb{P}\left(\int_0^t e^{\xi^1_{s-}} d\eta^1_s \le x^1, \ldots, \int_0^t e^{\xi^n_{s-}} d\eta^n_s \le x^n \right).$$

To this end, let us introduce an n-dimensional generalized Ornstein–Uhlenbeck process $X = (X^1, \ldots, X^n)$ defined by

$$X^i_t = e^{-\xi^i_t} \left(X^i_0 - \int_0^t e^{\xi^i_{s-}} d\eta^i_s \right)$$

for each $i = 1, \ldots, n$. The process X has two important properties. The first is the trivial equality

$$\{X^i_t \ge 0\} = \left\{ \int_0^t e^{\xi^i_{s-}} d\eta^i_s \le X^i_0 \right\}.$$

The second property is that X is a homogeneous Markov process as seen by the identity

$$X^i_{t+u} = e^{-(\xi^i_{t+u} - \xi_t)} \left(X^i_t - \int_t^{t+u} e^{(\xi^i_{s-} - \xi^i_t)} d\eta^i_s \right),$$

and the independence and stationarity of the increments of a Lévy process. Define the Markov semigroup $(P_t)_{t \ge 0}$ by

$$(P_t f)(x) = \mathbb{E}[f(X_t) | X_0 = x]$$

for bounded measurable f, and let $\mathcal{L}$ denote its generator.

These two properties combine to give a useful characterization of the joint distribution function:

$$F(t, x) = \mathbb{P}(X^1_t \ge 0, \ldots, X^n_t \ge 0 | X_0 = x) = (P_t \mathbb{1}_{[0,\infty)^n})(x) \tag{1}$$

Now we consider the situation as $t \uparrow \infty$. Our main result is the following

Proposition 2. *Suppose that for each $i = 1, \ldots, n$, almost surely*
1. *$\xi^i_t \to -\infty$, and*
2. *$\int_0^t e^{\xi^i_{s-}} d\eta^i_s$ converges to a finite limit.*

Then

$$F(x) = \mathbb{P}\left(\int_0^\infty e^{\xi^1_{s-}} d\eta^1_s \le x^1, \ldots, \int_0^\infty e^{\xi^n_{s-}} d\eta^n_s \le x^n \right) \tag{2}$$

if and only if F is the distribution function of a probability measure on $\mathbb{R}^n$ and $\mathcal{L}F = 0$.

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Corollary 3. *If F is given by equation (2) and $\mathcal{L}$ is hypoelliptic, then F is infinitely differentiable.*

Proof of Proposition 2. First the easy direction: If F is given by equation (2) then

$$\begin{aligned} F(x) &= \mathbb{P}\left\{\int_t^\infty e^{(\xi^i_{s-}-\xi^i_t)}d\eta^1_s \le e^{-\xi^i_t}\left(x^i - \int_0^t e^{\xi^i_{s-}}d\eta^i_s\right) \text{ for all } i=1,\dots,n\right\} \\ &= \mathbb{E}[F(X_t)|X_0=x] \end{aligned}$$

by the stationarity and independence of the increments of Lévy processes. Hence $P_tF = F$ which implies $\mathcal{L}F = 0$.

For the other direction, let $x_0 \in \mathbb{R}^n$ be such that

$$\mathbb{P}\left(\int_0^\infty e^{\xi^i_s}d\eta^i_s = x^i_0\right) = 0$$

for all $i = 1,\dots,n$. Note that the set of all such x_0's is dense in $\mathbb{R}^n$.

Now, consider the Markov process X initialized at $X_0 = x_0$. Note that $|X^i_t| \to \infty$ almost surely for each i by assumptions (1) and (2). Letting $Z_t = F(X_t)$, we see Z converges almost surely to $\mathbb{1}_{\{X^1_t\to\infty,\dots,X^n_t\to\infty\}}$ by the assumption that F is the distribution function of a probability measure. Furthermore, Z is a martingale as F is bounded and $\mathcal{L}$-harmonic. Hence

$$\begin{aligned} F(x_0) &= \lim_{t\uparrow\infty} \mathbb{E}[F(X_t)] \\ &= \mathbb{P}(X^1_t \to \infty,\dots,X^n_t\to\infty) \\ &= \mathbb{P}\left(\int_0^\infty e^{\xi^1_{s-}}d\eta^1_s \le x^1_0,\dots,\int_0^\infty e^{\xi^n_{s-}}d\eta^n_s \le x^n_0\right) \end{aligned}$$

by the bounded convergence theorem. The claim now follows from the observation that a distribution function is specified by its values on a dense set of points. $\square$

Remark 1. Carmona, Petit, and Yor have derived this result in the cases where $n = 1$ and either $\eta_t = t$ [2] or ξ and η are independent [3], by calculating the invariant distibution of the generalized Ornstein–Uhlenbeck process U defined by

$$U_t = e^{\xi_t}\left(u + \int_0^t e^{-\xi_{s-}}d\eta_s\right).$$

Their key observation is

$$\mathbb{P}\left(\int_0^t e^{\xi_{s-}}d\eta_s \le x\right) = \mathbb{P}(U_t \le x|U_0 = 0)$$

which is in the spirit of our equation (1). Donati-Martin, Ghomrasni, and Yor [4] use the same technique in the multi-dimensional setting.

The key difference between these approaches and ours is that we are concerned with the transition probabilities of our process X from x to 0, as opposed to the transition of U from 0 to x.

We are now ready to treat Dufresne's result.

Example 1. Let W be a standard Wiener process and c a positive constant. The corresponding generalized Ornstein–Uhlenbeck process

$$X_t = e^{-W_t+ct}\left(X_0 - \int_0^t e^{W_s-cs}ds\right)$$

satisfies the SDE

$$dX_t = \left[\left(c+\frac{1}{2}\right)x - 1\right]dt - X_t dW_t$$

and hence has generator

$$\mathcal{L} = \left[\left(c+\frac{1}{2}\right)x - 1\right]\frac{\partial}{\partial x} + \frac{x^2}{2}\frac{\partial^2}{\partial x^2}.$$

But since

$$F(x) = \int_0^x \frac{2^{2c}}{\Gamma(2c)}\frac{e^{-2/y}}{y^{1+2c}}dy, \quad x > 0$$

satisfies $\mathcal{L}F = 0$, we have proven Dufresne's theorem.

We now give another Brownian example:

Example 2. Let W and B be independent Wiener processes, c a positive constant, a a real constant. The random variable $\int_0^\infty e^{W_s - cs}(a\ ds + dB_s)$ has a density g given by

$$g(x) = C \frac{e^{2a \tan^{-1} x}}{(1+x^2)^{c+1/2}}$$

where $C > 0$ is such that $\int_{-\infty}^{\infty} g(x)dx = 1$. In this case, the relevant process X is given by

$$X_t = e^{-W_t + ct}\left(X_0 - \int_0^t e^{W_s - cs}(a\ ds + dB_s)\right)$$

which satisfies the SDE

$$dX_t = \left[\left(c + \frac{1}{2}\right)x - a\right]dt - X_t dW_t - dB_t.$$

We simply check that the generator of X is given by

$$\mathcal{L} = \left[\left(c + \frac{1}{2}\right) - a\right]\frac{\partial}{\partial x} + \frac{1}{2}(x^2+1)\frac{\partial^2}{\partial x^2}$$

and that $\mathcal{L}G = 0$ where $G(x) = \int_{-\infty}^{x} g(x)dx$. This example also appears in the paper of Carmona, Petit, and Yor [3].

We note in passing that we can handle the more general case $\int_0^\infty e^{W_s - cs}(a\ ds + dZ_s)$ where W and Z are Wiener processes with correlation ρ, by appealing to the identity

$$\begin{aligned}\int_0^\infty e^{W_s - cs}(a\ ds + dZ_s) &= \int_0^\infty e^{W_s - cs}(a\ ds + \rho dW_s + \sqrt{1-\rho^2}dB_s) \\ &= -1 + \int_0^\infty e^{W_s - cs}\left[\left(a + \rho c - \frac{\rho}{2}\right)ds + \sqrt{1-\rho^2}dB_s\right]\end{aligned}$$

which follows from the limit $\int_0^t e^{W_s - cs}[dW_s + (\frac{1}{2} - c)ds] = e^{W_t - ct} - 1 \to -1$ almost surely.

Remark 2. Several other proofs of Dufresne's result have appeared in the literature, in addition to the Carmona, Petit, and Yor papers [2, 3] mentioned above. The following list is not exhaustive, but merely indicates the various techniques that have been proposed. For instance, Yor [8] has shown that the distibution of the Dufresne integral is equal to the distribution of the first passage time of a Bessel process via Lamperti's relation. Matsumoto and Yor [7] recover the distribution from inverting a Laplace transform. Finally, Bailleul [1] has shown directly that if the Dufresne integral has a smooth density function f, then f must satisfy a certain ODE. The difficult part of this last proof is the verification, via Malliavin calculus, of the existence of this smooth density. The proof given here bypasses this technicality. It is inspired by an observation of Goodman [6] in the $c = 1/2$ case.

We now describe describe the generator of the process X in the general case. Suppose the characteristic function of (ξ_t, η_t) is given by

$$\mathbb{E}(e^{iu\cdot\xi_t + iv\cdot\eta_t}) = e^{t\psi(u,v)}$$

where by the Lévy–Khintchine formula ψ is of the form

$$\psi(u,v) = ia\cdot u + ib\cdot v - \frac{1}{2}u\cdot Au - u\cdot Bv - \frac{1}{2}v\cdot Cv + \int_{\mathbb{R}^n\times\mathbb{R}^n}(e^{iu\cdot p + iv\cdot q} - 1 - i(u\cdot p + v\cdot q)\mathbb{1}_{\{|p|+|q|<1\}})\nu(dp,dq)$$

where a and b are vectors in $\mathbb{R}^n$, A, B, and C are $n\times n$ matrices such that the matrix $\begin{pmatrix} A & B \\ B^{\mathrm{T}} & C\end{pmatrix}$ is symmetric and non-negative definite, and ν is a Borel measure such that $\int_{\mathbb{R}^n\times\mathbb{R}^n} 1\wedge(|p|+|q|)\nu(dp,dq) < \infty$.

Then the generator of the Markov semigroup is the integro-differential operator defined by

$$\begin{aligned}\mathcal{L}f(x) &= \sum_{i=1}^{n}\left[\left(\frac{1}{2}A_{ii}-a_i\right)x_i - b_i\right]\frac{\partial f(x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}(x_ix_jA_{ij}+2x_iB_{ij}+C_{ij})\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \\ &\quad + \int_{\mathbb{R}^n\times\mathbb{R}^n}\left[f(\exp(-p)x-q)-f(x)+\sum_{i=1}^{n}(x_ip_i+q_i)\frac{\partial f(x)}{\partial x_i}\mathbb{1}_{\{|p|+|q|<1\}}\right]\nu(dp,dq)\end{aligned}$$

where $\exp(-p)$ in the integral denotes the diagonal matrix with ith diagonal component e^{-p_i}.

We conclude this note by giving sufficient conditions for the hypotheses of Proposition 2 to hold:

Proposition 4. *Let ξ be a scalar Lévy process with triple (a,σ^2,μ). If*

$$\int_{\{|p|\geq 1\}}|p|\ \mu(dp)<\infty \quad \text{and } a+\int_{\{|p|\geq 1\}}p\ \nu(dp,dq)<0$$

then $\xi_t \to -\infty$ almost surely. If in addition η is a Lévy process with Lévy measure ν such that

$$\int_{\{|q|<1\}}|q|\ \nu(dq)<\infty \quad \text{and} \int_{|q|\geq 1}\log|q|\ \nu(dq)<\infty$$

then $\int_0^t e^{\xi_{s-}}d\eta_s$ converges almost surely.

Proof. The first condition is sufficient for $\mathbb{E}(|\xi_1|)<\infty$ and $\mathbb{E}(\xi_1)<0$. Since $\frac{1}{t}\xi_t\to\mathbb{E}(\xi_1)$ almost surely by the strong law of large numbers, it follows that there exists a random time $T<\infty$ such that $\xi_t<\frac{1}{2}\mathbb{E}(\xi_1)$ for all $t\geq T$. This proves $\xi_t\to-\infty$.

Now, if $\int_{\{|q|<1\}}|q|\ \nu(dq)<\infty$ then η can be decomposed into the sum of a Brownian motion with drift and a pure jump process

$$\eta_t = bt + cW_t + \tau_t^+ - \tau_t^-$$

for subordinators τ^+ and τ^-. The integral $\int_0^\infty e^{\xi_s}ds$ exists since

$$\int_0^\infty e^{\xi_s}ds < \int_0^T e^{\xi_s}ds + \int_T^\infty e^{-rs}ds < \infty$$

almost surely, where $r=\frac{1}{2}\mathbb{E}(\xi_1)$. Similarly, the Itô integral $\int_0^\infty e^{\xi_s}dW_s$ exists since $\int_0^\infty e^{2\xi_s}ds<\infty$ almost surely. Now, we need only show $\int_0^\infty e^{\xi_s}d\tau_s<\infty$ almost surely for a pure-jump subordinator τ. Since τ is increasing, the integral is a path-wise Lebesgue–Stieltjes integral. Hence, it suffices to show $I_t=\int_0^t e^{-rs}d\tau_s$ converges almost surely to a finite limit. But since $t\mapsto I_t(\omega)$is non-decreasing, the limit I_∞ always exists as a random variable valued in $[0,\infty]$. We now show that $(I_t)_{t\geq 0}$ converges in distribution to a finite-valued random variable.

Letting $\phi(u)=\int_0^\infty(e^{iuq}-1)\nu(dq)$ then

$$\mathbb{E}(e^{iuI_t}) = e^{\int_0^t\phi(ue^{-rs})ds} = e^{\frac{1}{r}\int_{e^{-rt}}^1\frac{1}{x}\phi(ux)dx}$$

Taking $u>0$ without loss, we have the computation

$$\begin{aligned}\int_0^1\frac{1}{x}|\phi(ux)|dx &\leq \int_0^\infty\int_0^1\left(\frac{2}{x}\right)\wedge(uq)\ dx\ \nu(dq) \\ &= \int_0^\infty[(uq)\wedge 2+2\log^+(uq/2)]\nu(dq)\end{aligned}$$

where we have used the bound $|e^{iz}-1|<2\wedge|z|$. The expression on the last line is finite for all $u>0$ and converges to 0 as $u\downarrow 0$. Therefore, the characteristic function of I_t converges pointwise to

$$g(u) = e^{\frac{1}{r}\int_0^1\frac{1}{x}\phi(ux)dx}.$$

Since g is continuous at $u=0$, Lévy's continuity theorem implies $(I_t)_{t\geq 0}$ converges in distribution. □

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