PARALLEL SHIFTS OF AT-THE-MONEY IMPLIED VOLATILITY

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Abstract. Suppose the at-the-money implied volatility term structure can only move by parallel shifts, in the sense that $\Sigma_t(\tau) = \Sigma_0(\tau) + \xi_t$ a.s. for all (t, τ) and some process ξ. Then $\xi_t = 0$ a.s. for all t, under a mild technical assumption on the underlying stock price. As a by-product of the proof, a Dybvig–Ingersoll–Ross-type theorem for long implied volatility is presented.

1. Introduction

Let $\Sigma_t(\tau)$ be the at-the-money (ATM) implied volatility of a call option at time $t \geq 0$ with time until maturity $\tau > 0$. If the ATM implied volatility term structure only moves by parallel shifts, that is, if

$$\Sigma_t(\tau) = \Sigma_0(\tau) + \xi_t$$

for all (t, τ) for some process ξ, then must it be the case that $\xi_t = 0$ for all $t \geq 0$?

To put this question in context, consider the analogous problem in interest rate theory. Let $y_t(\tau)$ be the yield of a zero-coupon bond, and suppose that

$$y_t(\tau) = y_0(\tau) + \eta_t$$

for some process η. Must $\eta_t = 0$ a.s. for all $t \geq 0$? The answer here is no. For instance, suppose the spot interest rate process r is a Lévy process and the yield is given by the formula

$$y_t(\tau) = -\frac{1}{\tau}\mathbb{E}[e^{-\int_t^{t+\tau} r_s \, ds} | \mathcal{F}_t]$$

where $(\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by the process r. Then a quick calculation yield the identity

$$y_t(\tau) = y_0(\tau) + r_t - r_0.$$

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The author and Chris Rogers [7] have previously considered the problem, proposed to us by Steve Ross, of whether the whole implied volatility surface could only move by parallel shifts; that is, does

$$\Sigma_t(\tau, k) = \Sigma_0(\tau, k) + \xi_t$$

for all (t, τ, k) imply $\xi_t = 0$ a.s. for all $t \geq 0$? Here $\Sigma_t(\tau, k)$ is the implied volatility at time $t \geq 0$ of a call option with time to maturity $\tau > 0$ and log-moneyness $k \in \mathbb{R}$, so that the ATM implied volatility is given by $\Sigma_t(\tau) = \Sigma_t(\tau, 0)$. We found that the answer is yes if, for instance, there exists a $p \in (0, 1)$ such that the map $t \to -\log \mathbb{E}(S_t^p)$ is sub-linear, where S is the price of the underlying stock.

In contrast to the above discussion, it was also shown in [7] that it is possible for the implied average variance surface $\Sigma_t(\tau, k)^2$ to move only by parallel shifts. Indeed, consider stock dynamics of the form

$$dS_t = S_t f(t) dW_t$$

where f is a deterministic function of time. Then a routine calculation shows

$$\Sigma_t(\tau, k)^2 = \frac{1}{\tau} \int_t^{t+\tau} f(s)^2 ds$$

If $f(t) = t$ for all $t \geq 0$, then

$$\Sigma_t(\tau, k)^2 = \frac{1}{2\tau}[(t + \tau)^2 - t^2] = \Sigma_0(\tau, k)^2 + t$$

for all (t, τ, k).

In this note, we show that the ATM implied volatility term structure cannot move only by parallel shifts, subject to a technical condition on the dynamics of the price of the underlying stock. In particular, in most models of interest, including virtually all local and stochastic volatility models, the implied volatility must move at different rates over the length of term structure. The lesson is that naive modelling of the implied volatility term structure — for instance, assuming that it moves only by parallel shifts — may introduce arbitrage.

A main tool of the analysis is a theorem on the monotonicity of long implied volatility. The first theorem of this type, proven by Dybvig, Ingersoll, and Ross [3], says that the long rate, defined as the limit of the zero-coupon bond yield as the maturity goes to infinity, is non-decreasing. There has been recent interest in this theorem. Hubalek, Klein, and Teichmann [6] provided a short proof of the Dybvig–Ingersoll–Ross result under the assumption of the existence of an equivalent martingale measure. See the pre-prints of Schulze [8] and Goldammer and Schmock [4] for a discussion of the relation of the Dybvig–Ingersoll–Ross result to various notions of no-arbitrage. The version of the theorem for the asymptotic difference of long implied volatilities given in our Theorem 3.1 is most closely related to the work of Kardaras and Platen [5].

The paper is structured as follows: In section 2, the assumptions, notation, and main results are presented. In section 3, the connection between the dynamics of the implied volatility term structure and the Dybvig–Ingersoll–Ross theorem of interest rate theory is explored. Finally, the main results are proved in section 4.

2. The set-up and main results

Let $S = (S_t)_{t\geq 0}$ be a positive martingale defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ is trivial. Without loss of generality, we will assume $S_0 = 1$.

The process S models the evolution, under a fixed risk-neutral measure $\mathbb{P}$, of the price of a given stock in an economy where the risk-free interest rate is identically zero. The case where interest rates are time-varying but deterministic can easily be handled in this framework by passing to discounted prices, but a model with fully stochastic interest rates is outside of the scope of this paper.

Recall that in this setting, the Black–Scholes implied volatility at time $t \geq 0$ of a European call option with log-moneyness $k \in \mathbb{R}$ and time to maturity $\tau > 0$ is defined as the unique non-negative solution $\sigma = \Sigma_t(\tau, k)$ of the equation

$$\mathbb{E}\left[\left(\frac{S_{t+\tau}}{S_t} - e^k\right)^+ |\mathcal{F}_t\right] = \mathrm{BS}(\tau\sigma^2, k)$$

where the Black–Scholes call price function is defined by

$$\mathrm{BS}(v,k) = \begin{cases} \Phi\left(-\frac{k}{\sqrt{v}} + \frac{\sqrt{v}}{2}\right) - e^k \Phi\left(-\frac{k}{\sqrt{v}} - \frac{\sqrt{v}}{2}\right) & \text{if } v > 0 \\ (1-e^k)^+ & \text{if } v = 0. \end{cases}$$

where Φ is the standard normal distribution function. Let $\Sigma_t(\tau) = \Sigma_t(\tau, 0)$ be the at-the-money Black–Scholes implied volatility. It is easy to see that in this case we have the explicit formula

$$\Sigma_t(\tau) = -\frac{2}{\sqrt{\tau}}\Phi^{-1}\left(\frac{1}{2}\mathbb{E}\left[\frac{S_{t+\tau}}{S_t} \wedge 1 |\mathcal{F}_t\right]\right).$$

We are now ready to state the main results of this note. The proofs of these results are deferred to section 4.

Theorem 2.1. *Suppose the ATM implied volatility only moves by parallel shifts, so that the equation*

$$\Sigma_t(\tau) = \Sigma_0(\tau) + \xi_t$$

holds for all (t, τ) and some adapted process ξ. If

(1)

$$\limsup_{T\uparrow\infty} \frac{\sqrt{-\log \mathbb{E}(S_T \wedge 1|\mathcal{F}_t)} - \sqrt{-\log \mathbb{E}(S_T \wedge 1)}}{\sqrt{T}} = 0 \text{ a.s. for all } t \geq 0,$$

then $\xi_t = 0$ a.s. for all $t \geq 0$.

Remark 1. If the time index set is countable, then the left-hand side of equation (1) defines a random variable, that is, a measurable function on the sample space Ω. However, if the time index set is uncountable, there is no such guarantee. Therefore, we make the following convention throughout: the notation sup and inf will denote the essential supremum and infimum respectively, and hence the left-hand side of equation (1) should be interpreted as

$$\limsup_{T\uparrow\infty} = \text{ess inf}_{T>0} \text{ ess sup}_{T'>T}.$$

Theorem 2.1 may seem of limited interest on its face, since it is not immediately clear how to verify equation (1). However, it turns out that it can be applied to a large class of models. First, we note that Theorem 2.1 applies to all models where the increments of the logarithmic stock are independent.

Theorem 2.2. *If $\log S$ has independent increments then equation (1) holds.*

The next theorem shows that, in fact, the Theorem 2.1 actually applies to virtually all local and stochastic volatility models.

Theorem 2.3. *Suppose S is a continuous martingale with dynamics*

$$dS_t = S_t \sigma_t dW_t$$

where W is a Brownian motion. Suppose that the positive volatility process σ is either

(LV) *of the form $\sigma_t = A(t, S_t)$ for a smooth function A or*
(SV) *has dynamics*

$$d\sigma_t = \alpha(t, \sigma_t)dt + \beta(t, \sigma_t)dZ_t$$

for another Brownian motion Z and smooth functions α and β.

If the ATM implied volatility only moves by parallel shifts, then $\sigma_t = \sigma_0$ a.s. for all $t \geq 0$; i.e. S is a geometric Brownian motion.

3. A Dybvig–Ingersoll–Ross-type theorem

In this section, we explore the connection between equation (1) and the result of Dybvig, Ingersoll, and Ross [3] that says that long yields on zero-coupon bonds can never fall.

Under the assumptions of the last section, we will prove the following Dybvig–Ingersoll–Ross-type theorem:

Theorem 3.1. *The long implied volatility never falls in the sense that the inequality*

$$\liminf_{\tau\uparrow\infty} \Sigma_t(\tau, k_1) - \Sigma_s(\tau, k_2) \geq 0 \;\; a.s.$$

holds for all $k_1, k_2 \in \mathbb{R}$ *and* $0 \leq s \leq t$.

To prove Theorem 3.1 we need several lemmas.

Lemma 3.2. *Let* $(X_n)_n$ *be a sequence of non-negative random variables. Then*

$$\limsup_{n\uparrow\infty} X_n^{1/n} \leq \limsup_{n\uparrow\infty} [\mathbb{E}(X_n)]^{1/n} \quad a.s.$$

Proof. Let $(Y_n)_n$ be a sequence of non-negative random variables with $\mathbb{E}(Y_n) = 1$ for all n. By Markov's inequality we have

$$\mathbb{P}(Y_n^{1/n} > 1+\epsilon) = \mathbb{P}[Y_n > (1+\epsilon)^n] \leq (1+\epsilon)^{-n}$$

for any $\epsilon > 0$. Since the right hand side of above inequality is summable, it follows that

$$\mathbb{P}(Y_n^{1/n} > 1+\epsilon \text{ infinitely often}) = 0$$

by the first Borel–Cantelli lemma, and in particular by letting $\epsilon \downarrow 0$ through a countable sequence,

$$\limsup_{n\uparrow\infty} Y_n^{1/n} \leq 1 \quad \text{a.s.}$$

Now, there is no loss assuming $\mathbb{E}(X_n)$ is strictly positive but finite for all n. Note that for every $\delta > 0$, we have the inequality $[\mathbb{E}(X_m)]^{1/m} \leq (1+\delta) \limsup_{n\uparrow\infty} [\mathbb{E}(X_n)]^{1/n}$ for m large enough. It follows that

$$\limsup_{m\uparrow\infty} \left(\frac{X_m}{\mathbb{E}(X_m)} \right)^{1/m} \geq (1+\delta)^{-1} \frac{\limsup_{m\uparrow\infty} X_m^{1/m}}{\limsup_{n\uparrow\infty} [\mathbb{E}(X_n)]^{1/n}}$$

The result now follows from letting $Y_n = X_n / \mathbb{E}(X_n)$ in the above and sending $\delta \downarrow 0$. $\square$

Lemma 3.3. *For all* $t \geq 0, \tau > 0$, *there exists a constant* $0 < C \leq 2$ *such that the following bounds hold almost surely:*

(1) $-C \leq \sqrt{\tau}\Sigma_t(\tau) - \sqrt{-8\log\mathbb{E}\left(\frac{S_{t+\tau}}{S_t}\wedge 1\middle|\mathcal{F}_t\right)} \leq 0,$

(2) $S_t \wedge 1 \leq \frac{\mathbb{E}(S_{t+\tau}\wedge 1|\mathcal{F}_t)}{\mathbb{E}\left(\frac{S_{t+\tau}}{S_t}\wedge 1|\mathcal{F}_t\right)} \leq S_t \vee 1,$ *and*

(3) $-C - \sqrt{8(\log S_t)^-} \leq \sqrt{\tau}\Sigma_t(\tau) - \sqrt{-8\log\mathbb{E}\left(S_{t+\tau}\wedge 1\middle|\mathcal{F}_t\right)} \leq \sqrt{8(\log S_t)^+}$

(Numerical evidence suggests that the smallest such C is approximately 1.03.*)*

Proof. (1) It is enough to show there is a finite constant $c > 0$ such that

$$0 \leq \sqrt{-2\log y} + \Phi^{-1}(y/2) \leq c$$

for all $0 < y \leq 1$, or equivalently, letting $R(x) = e^{x^2/2}\int_x^\infty e^{-s^2/2}ds$ denote the Gaussian Mills' ratio,

$$0 \leq \sqrt{x^2 - 2\log[R(x)/R(0)]} - x \leq c$$

for all $x \geq 0$, and to then take $C = 2c$.

It is well known that

$$R(x) \leq e^{x^2/2}\int_x^\infty \frac{s}{x}e^{-s^2/2}ds = \frac{1}{x}$$

for $x > 0$. Since

$$R'(x) = xR(x) - 1 \leq 0$$

we see that R is decreasing on $[0,\infty)$ and hence

$$R(x) \leq R(0).$$

This shows $\sqrt{x^2 - 2\log[R(x)/R(0)]} \geq x$.

The existence of the finite upper bound $c > 0$ follows from the fact that the function $f(x) = \sqrt{x^2 - 2\log[R(x)/R(0)]} - x$ is continuous and $f(0) = 0 = \lim_{x\to\infty} f(x)$. Numerical evidence suggests $c \approx 0.515$, but the following argument shows $c \leq 1$. Birnbaum [1] noted that the Cauchy–Schwarz inequality implies

$$\left(\int_x^\infty e^{-s^2/2}ds\right)\left(\int_x^\infty s^2 e^{-s^2/2}ds\right) \geq \left(\int_x^\infty s e^{-s^2/2}ds\right)^2$$

which translates to

$$R(x)^2 + xR(x) \geq 1 \Rightarrow R(x) \geq \frac{1}{2}(\sqrt{4+x^2} - x) = \frac{2}{\sqrt{4+x^2}+x} \geq \frac{1}{x+1}.$$

Hence

$$(e^x R(x))' = e^x[(x+1)R(x) - 1] \geq 0$$

so that

$$R(x) \geq R(0)e^{-x}.$$

Hence

$$\sqrt{x^2 - 2\log[R(x)/R(0)]} \leq \sqrt{x^2+2x} < x+1.$$

(2) This follows by noting the elementary inequality

$$\left(\frac{S_{t+\tau}}{S_t} \wedge 1\right)(S_t \wedge 1) \leq S_{t+\tau} \wedge 1 \leq \left(\frac{S_{t+\tau}}{S_t} \wedge 1\right)(S_t \vee 1),$$

and taking the conditional expectation with respect to $\mathcal{F}_t$.

(3) Combining (1) and (2) with the inequality $\sqrt{(a-b)^+} \geq \sqrt{a} - \sqrt{b}$ which holds for all $a, b \geq 0$, yields

$$\begin{aligned}
\sqrt{-\log \mathbb{E}(S_{t+\tau} \wedge 1)} &\geq \sqrt{\left[-\log \mathbb{E}\left(\frac{S_{t+\tau}}{S_t} \wedge 1\right) - \log S_t \vee 1\right]^+} \\
&\geq \sqrt{-\log \mathbb{E}\left(\frac{S_{t+\tau}}{S_t} \wedge 1\right)} - \sqrt{\log S_t \vee 1} \\
&\geq \sqrt{\tau}\Sigma_t(\tau)/\sqrt{8} - \sqrt{(\log S_t)^+}.
\end{aligned}$$

Similary, the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ yields

$$\begin{aligned}
\sqrt{-\log \mathbb{E}(S_{t+\tau} \wedge 1)} &\leq \sqrt{-\log \mathbb{E}\left(\frac{S_{t+\tau}}{S_t} \wedge 1\right) - \log S_t \wedge 1} \\
&\leq \sqrt{-\log \mathbb{E}\left(\frac{S_{t+\tau}}{S_t} \wedge 1\right)} + \sqrt{-\log S_t \wedge 1} \\
&\leq C + \sqrt{\tau}\Sigma_t(\tau)/\sqrt{8} + \sqrt{(\log S_t)^-}.
\end{aligned}$$

□

Proof of Theorem 3.1. In [7], it was shown that the implied volatility surface flattens at long maturities in the sense

$$\lim_{\tau \uparrow \infty} \sup_{k_1, k_2 \in [-M, M]} |\Sigma_t(\tau, k_1) - \Sigma_t(\tau, k_2)| = 0 \text{ a.s.}$$

for all $M > 0$ and $t \geq 0$. Therefore, it is enough consider the case where $k_1 = k_2 = 0$.

By Lemma 3.2 we have

$$\liminf_{T \uparrow \infty} -\frac{1}{T}\log \mathbb{E}(S_T \wedge 1 | \mathcal{F}_t) + \frac{1}{T}\log \mathbb{E}(S_T \wedge 1 | \mathcal{F}_s) = -\limsup_{T \uparrow \infty} \frac{1}{T} \log \frac{\mathbb{E}(S_T \wedge 1|\mathcal{F}_t)}{\mathbb{E}(S_T \wedge 1|\mathcal{F}_s)} \geq 0 \text{ a.s.}$$

for all $0 \leq s \leq t$.

The real analysis fact that for non-negative sequences $(a_n)_n$ and $(b_n)_n$,

$$\liminf_n a_n - b_n \geq 0 \Rightarrow \liminf_n \sqrt{a_n} - \sqrt{b_n} \geq 0$$

together with Lemma 3.3 (3) implies

$$\liminf_{\tau\uparrow\infty} \Sigma_t(\tau) - \sqrt{1+(t-s)/\tau}\Sigma_s(t-s+\tau) \geq 0$$

all $0 \leq s \leq t$.

Since S is a martingale, the conditional version of Jensen's inequality implies $\tau \mapsto \mathbb{E}[S_{t+\tau} \wedge 1|\mathcal{F}_t]$ is decreasing almost surely for each $t \geq 0$, and hence $\tau \mapsto \sqrt{\tau}\Sigma_t(\tau)$ is increasing. This observation implies

$$\sqrt{1+(t-s)/\tau}\Sigma_s(t-s+\tau) \geq \Sigma_s(\tau)$$

almost surely for all $0 \leq s \leq t$ and $\tau > 0$, concluding the proof. □

Remark 2. Using the ideas of Hubalek, Klein, and Teichman [6], the weaker inequality

$$\limsup_{\tau\uparrow\infty} \Sigma_t(\tau, k_1) - \Sigma_s(\tau, k_2) \geq 0 \text{ a.s.}$$

was shown in [7] to hold for all all $k_1, k_2 \in \mathbb{R}$ and $0 \leq s \leq t$. The same method of proof used there also shows the following inequality

$$\limsup_{\tau\uparrow\infty} \Sigma_t(\tau, k_1) \geq \limsup_{\tau\uparrow\infty} \Sigma_s(\tau, k_2) \text{ a.s.}$$

Above inequality is similar to the formulation of the Dybvig–Ingersoll–Ross theorem in [4]. On the other hand, the same method of proof used for Theorem 3.1 shows

$$\liminf_{\tau\uparrow\infty} \Sigma_t(\tau, k_1) \geq \liminf_{\tau\uparrow\infty} \Sigma_s(\tau, k_2) \text{ a.s.}$$

This formulation resembles the formulation in [5].

We now address the mysterious technical condition given by equation (1) in Theorem 2.1. Indeed, Lemma 3.3 (3) shows

$$\limsup_{T\uparrow\infty} \frac{\sqrt{-\log \mathbb{E}(S_T \wedge 1|\mathcal{F}_t)} - \sqrt{-\log \mathbb{E}(S_T \wedge 1)}}{\sqrt{T}} = \limsup_{\tau\uparrow\infty} \Sigma_t(\tau) - \sqrt{1+t/\tau}\Sigma_0(t+\tau)$$

Hence, we have the following useful lemma:

Lemma 3.4. *Equation (1) is equivalent to*

$$\limsup_{\tau\uparrow\infty} \Sigma_t(\tau) - \sqrt{1+t/\tau}\Sigma_0(t+\tau) = 0 \text{ a.s.}$$

for all $t \geq 0$.

Remark 3. Note that the proof of Theorem 3.1 already gives the bound

$$\liminf_{\tau\uparrow\infty} \Sigma_t(\tau) - \sqrt{1+t/\tau}\Sigma_0(t+\tau) \geq 0 \text{ a.s.}$$

4. The proofs of the main results

In this section, we prove the results of Section 2.

Proof of Theorem 2.1. Suppose the implied volatility term structure moves only by parallel shifts and assume that equation (1) holds.

Let $F(n) = \sqrt{n}\Sigma_0(n)$ so that

$$F(m+n) = F(m) + \sqrt{m}\xi_n - a_n(m)$$

where

$$a_n(m) = \sqrt{m}\Sigma_n(m) - \sqrt{m+n}\Sigma_0(m+n).$$

Letting $m = ni$ and summing over i yields

$$F(kn) = \sum_{i=1}^{k} \sqrt{in}\xi_n - \sum_{i=1}^{k} a_n(ni).$$

Now by Lemma 3.4 we have

$$m^{-1/2}a_n(m) \to 0$$

as $m \uparrow \infty$ and hence

$$k^{-3/2}\sum_{i=1}^{k} a_n(ni) \to 0$$

Since $k^{-3/2}\sum_{i=1}^{k}\sqrt{i} \to \frac{2}{3}$ we have

$$\lim_{m\to\infty} \frac{F(m)}{m^{3/2}} = \lim_{m\to\infty} \frac{\Sigma_0(m)}{m} = \frac{2}{3}\frac{\xi_n}{n}.$$

That is, there is a non-random constant $\ell \geq 0$ such that $\xi_n = \frac{3}{2}n\ell$. It remains to show $\ell = 0$.

In particular, ξ_n is not random for all $n \geq 0$ and hence

$$\mathbb{E}\left(\frac{S_{m+n}}{S_n} \wedge 1 | \mathcal{F}_n\right) = \mathbb{E}\left(\frac{S_{m+n}}{S_n} \wedge 1\right)$$

Lemma 3.3(2) yields

$$\mathbb{E}\left(\frac{S_{m+n}}{S_n} \wedge 1\right) \leq \frac{\mathbb{E}(S_{m+n} \wedge 1)}{\mathbb{E}(S_n \wedge 1)} \tag{2}$$

Let $G(m) = \sqrt{-8 \log \mathbb{E}(S_m \wedge 1)}$ so that $0 \le G(m) - F(m) \le C$ for all m and some constant $C > 0$, by Lemma 3.3. Then

$$\begin{aligned}\frac{3}{2}\sqrt{m} n\ell + G(m) &\ge \frac{3}{2} n\sqrt{m}\ell + F(m) \\ &= \sqrt{m}\xi_n + \sqrt{m}\Sigma_0(m) \\ &= \sqrt{m}\Sigma_n(m) \\ &\ge -C + \sqrt{-8\log\mathbb{E}\left(\frac{S_{m+n}}{S_n}\wedge 1\right)} \\ &\ge -C + \sqrt{G(m+n)^2 - G(n)^2}\end{aligned}$$

where the last line follows from equation (2).

Letting $m = n$ again, dividing by $n^{3/2}$, sending $n \uparrow \infty$, and using the limit $\lim_{n\uparrow\infty} n^{-3/2} G(n) = \ell$ yields

$$\frac{5}{2}\ell \ge \sqrt{7}\ell$$

so that $\ell = 0$ as desired. □

Proof of Theorem 2.2. If $\log S$ has independent increments then the conditional expectation $\mathbb{E}\left(\frac{S_T}{S_t}\wedge 1 | \mathcal{F}_t\right)$ is not random. As before, taking unconditional expectations of the bounds in Lemma 3.3(2) yields

$$\frac{S_t \wedge 1}{\mathbb{E}(S_t \vee 1)} \le \frac{\mathbb{E}(S_T \wedge 1|\mathcal{F}_t)}{\mathbb{E}(S_T \wedge 1)} \le \frac{S_t \vee 1}{\mathbb{E}(S_t \wedge 1)}$$

uniformly in $T > t$. The limit

$$\lim_{T\uparrow\infty} \frac{\log\mathbb{E}(S_T\wedge 1|\mathcal{F}_t) - \log\mathbb{E}(S_T \wedge 1)}{T} = 0 \text{ a.s. for all } t \ge 0$$

is now immediate. To conclude the proof, note that $a_n - b_n \to 0$ implies $\sqrt{a_n} - \sqrt{b_n} \to 0$ for non-negative sequences $(a_n)_n$ and $(b_n)_n$. □

Proof of Theorem 2.3. Since S is a continuous martingale with a continuous volatility process, it is well known that the ATM implied volatility converges as $\tau \downarrow 0$ to the spot volatility; see [2], for instance, for a proof. Now, if the implied volatility term structure only moves by parallel shifts, then we have

$$\xi_t = \Sigma_t(0) - \Sigma_0(0) = \sigma_t - \sigma_0.$$

On the other hand, the Dybvig–Ingersoll–Ross-type inequality

$$\xi_t - \xi_s = \liminf_{\tau\uparrow\infty} \Sigma_t(\tau) - \Sigma_s(\tau) \ge 0 \text{ a.s.}$$

of Theorem 3.1 holds for all $0 \leq s \leq t$. In particular, since ξ is non-decreasing, we must conclude that σ is of bounded variation, so the local martingale term in its semimartingale decomposition must vanish.

In the case (LV), we have

$$d\sigma_t = \left(\frac{\partial A}{\partial t} + \frac{\partial^2 A}{\partial S^2} S_t^2 \sigma_t^2 \right) dt + \frac{\partial A}{\partial S} S_t \sigma_t dW$$

and therefore $\frac{\partial A}{\partial S} = 0$ identically so that $A(t,S) = A(t,S_0)$ for all $S > 0$. In particular, the volatility $\sigma_t = A(t,S_0)$ is deterministic for all $t \geq 0$.

Similarly, in the case (SV), we conclude $\beta(t,\sigma_t) = 0$ a.s. for all $t \geq 0$. Hence, the process σ satisfies the ODE

$$d\sigma_t = \alpha(t,\sigma_t)dt.$$

Since $\mathcal{F}_0$ is trivial by assumption, the initial stock price S_0 is almost surely constant, and hence so is the volatility σ_t for each $t \geq 0$.

In both cases, then, $\log S$ has independent increments since

$$\log S_t = -\frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s.$$

Theorem 2.2 now says that S verifies equation (1) and hence the conclusion follows from Theorem 2.1. ☐

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