Optimization Michael Tehranchi

Transportation algorithm: Why does pivoting decrease the total cost?

Let $f(x) = \sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij}$ be the objective function for the transportation problem. Suppose x^0 is an initial feasible assigment, and x^1 the next feasible assignment after pivoting. Is it true that $f(x^1) < f(x^0)$?

First note that for any feasible x and any collection $(\lambda_i)_i$ and $(\mu_j)_j$ of Lagrange multipliers we have

$$\begin{aligned} f(x) &= \sum_{i=1}^m \sum_{j=1}^n d_{ij} x_{ij} + \sum_{i=1}^m \lambda_i \left(S_i - \sum_{j=1}^n x_{ij} \right) + \sum_{j=1}^n \mu_j \left(D_j - \sum_{i=1}^m x_{ij} \right) \\ &= \sum_{i=1}^m \sum_{j=1}^n (d_{ij} - \lambda_i - \mu_j) x_{ij} + \sum_{i=1}^m \lambda_i S_i + \sum_{j=1}^n \mu_j D_j. \end{aligned}$$

Now consider the initial feasible assignment x^0. Remember x^0 is a basic feasible solution of the supply and demand constraints. Let $B = \{(i,j) : x^0_{ij} > 0\}$ and $N = \{(i,j) : x^0_{ij} = 0\}$. The set B is the basis for the b.f.s x^0.

In the transportation algorithm we choose λ^0_i and μ^0_j such that $\lambda^0_i + \mu^0_j = d_{ij}$ for all $(i,j) \in B$. Hence

$$\begin{aligned} f(x^0) &= \sum_{(i,j)\in B} (d_{ij} - \lambda^0_i - \mu^0_j) x^0_{ij} + \sum_{(i,j)\in N} (d_{ij} - \lambda^0_i - \mu^0_j) x^0_{ij} + \sum_{i=1}^m \lambda^0_i S_i + \sum_{j=1}^n \mu^0_j D_j \\ &= \sum_{i=1}^m \lambda^0_i S_i + \sum_{j=1}^n \mu^0_j D_j \end{aligned}$$

Now consider the next feasible assignment x^1 (assumed non-degenerate). One of the initial non-basic cells (the pivot cell) is made basic, so that there is a cell $(i_N, j_N) \in N$ such that

$$x^0_{i_N,j_N} = 0 \text{ but } x^1_{i_N,j_N} = \epsilon > 0.$$

Hence

$$\begin{aligned} f(x^1) &= \sum_{(i,j)\in B} (d_{ij} - \lambda^0_i - \mu^0_j) x^1_{ij} + \sum_{(i,j)\in N} (d_{ij} - \lambda^0_i - \mu^0_j) x^1_{ij} + f(x^0) \\ &= (d_{i_N,j_N} - \lambda_{i_N} - \mu_{j_N})\epsilon + f(x^0). \end{aligned}$$

But since $d_{i_N,j_N} - \lambda_{i_N} - \mu_{j_N} < 0$ in the pivot cell, we have shown $f(x^1) < f(x^0)$.