Optimization Michael Tehranchi

Lecture 9 - Easter 2009

III. Two-person zero-sum games

- Player I chooses among m strategies.
- Player II chooses among n strategies.
- If Player I chooses strategy i and Player II chooses strategy j, then
  - Player I is paid £$a_{i,j}$, and
  - Player II is paid £$(-a_{i,j})$.
- The matrix $A = (a_{i,j})_{i,j}$ is called the *payoff* matrix of the game.

We think of these players playing this game over and over again. In most games, it is not a good idea to play the same strategy every time. Hence, we consider *mixed strategies*:

- Player I plays strategy i with probability p_i, and
- Player II plays strategy j with probability q_j.

A *pure* strategy is a mixed strategy that assigns probability 1 to one of the strategies and 0 to the rest.

The players' problems

- Player I's problem: maximize $\min_j \sum_{i=1}^m p_i a_{i,j}$ subject to $\sum_{i=1}^m p_i = 1$ and $p_i \geq 0$ for all $i = 1, \ldots, m$.
- Player II's problem: minimize $\max_i \sum_{j=1}^n a_{i,j} q_j$ subject to $\sum_{j=1}^m q_j = 1$ and $q_j \geq 0$ for all $j = 1, \ldots, n$.

Using the notation $e = (1, \ldots, 1)^{\mathrm{T}}$ for a column vector in $\mathbb{R}^m$ or $\mathbb{R}^n$ as context dictates, Player I's problem is

$$P: \text{ maximize } v \text{ subject to } A^{\mathrm{T}} p \geq ve, \; e^{\mathrm{T}} p = 1, \; p \geq 0.$$

Player II's problem is the dual

$$D: \text{ minimize } w \text{ subject to } Aq \leq we, \; e^{\mathrm{T}} q = 1, \; q \geq 0.$$

Theorem. If vectors $p^* \in \mathbb{R}^m$, $q^* \in \mathbb{R}^n$ and $v^* \in \mathbb{R}$ satisfy

- $A^{\mathrm{T}} p^* \geq v^* e, \; e^{\mathrm{T}} p^* = 1, \; p^* \geq 0$ (primal feasibility)
- $Aq^* \leq v^* e, \; e^{\mathrm{T}} q^* = 1, \; q^* \geq 0$ (dual feasibility)
- $v^* = (p^*)^{\mathrm{T}} A q^*$ (complementary slackness)

then the pair v^*, p^* is optimal for P, the pair v^*, q^* is optimal for D, and v^* is called the *value of the game*.

Proof: Let v, p be feasible for P and w, q be feasible for D. Then we have the usual weak duality calculation:

$$\begin{aligned} v &\leq v + q^{\mathrm{T}}(A^{\mathrm{T}} p - ve) \\ &= q^{\mathrm{T}} A^{\mathrm{T}} p \\ &= w + p^{\mathrm{T}}(Aq - we) \leq w \end{aligned}$$

Hence if v^*, p^* is feasible for P and v^*, q^* is feasible for D then $v \leq v^* \leq w$ proving the optimality of v^*, p^* and v^*, q^* for their respective problems. □

Finding optimal mixed strategies for games

(1) *Look for a saddle point.* If a saddle point (i, j) exists, then the optimal strategy for player I is the pure strategy i, and the optimal strategy for player II is the pure strategy j. The value of the game is $a_{i,j}$.

(2) *Look for dominating strategies.* A row i *dominates* row i' if

$$a_{i,j} \geq a_{i',j} \text{ for all } j = 1, \ldots, n$$

Player I should never play strategy i'. Similarly, a column j *dominates* column j' if

$$a_{i,j} \leq a_{i,j'} \text{ for all } i = 1, \ldots, m.$$

Player II never plays strategy j'.

(3) *Draw a picture.*

(4) *Find a solution satisfying the conditions of the theorem.*

(5) *Use the simplex algorithm.* If $\min_{i,j} a_{i,j} > 0$ we can put the problem in a form to use the simplex algorithm as follows:

Let $x = p/v$ so that problem P becomes

$$\text{maximize } v \text{ subject to } A^{\mathrm{T}}x \geq e, e^{\mathrm{T}}x = 1/v, x \geq 0$$

which is equivalent to

$$\text{minimize } e^{\mathrm{T}}x \text{ subject to } A^{\mathrm{T}}x \geq e, x \geq 0$$

We could use the two-phase method. Or we could look at the dual problem

$$\text{maximize } e^{\mathrm{T}}y \text{ subject to } A^{\mathrm{T}}y \leq e, y \geq 0$$

which is exactly in the form to use the one-phase method.