

OVERVIEW OF THE COURSE

Highlights

- General constrained optimization
 - The Lagrangian sufficiency theorem
 - Primal and dual problems
- Linear programming
 - The necessary and sufficient theorem for optimality
 - The simplex algorithm
- Applications to game theory
- Applications to networks
 - The max-flow min-cut theorem
 - The Ford–Fulkerson algorithm
 - The transportation algorithm

I. LAGRANGIAN METHODS

Typical problem

minimize $f(x)$ subject to $g(x) = b, \quad x \in X.$

- X is a given subset of \mathbb{R}^n .
- $f : X \rightarrow \mathbb{R}$ is the *objective function*.
- $g : X \rightarrow \mathbb{R}^m$ defines m functional constraints.
- Any $x \in X$ such that $g(x) = b$ is a *feasible solution*.
- x^* is *optimal* if x^* is feasible and $f(x^*) \leq f(x)$ for all feasible x .

Notation.

- \mathbb{R}^n is the space of n -dimensional *column* vectors
- $x = (x_1, \dots, x_n)^T$

A review of unconstrained optimization and convexity

Theorem. Suppose x^* is minimizes $f(x)$ subject to $x \in \mathbb{R}^n$. If f is differentiable at x^* , then $\nabla f(x^*) = 0$.

Theorem. Suppose that X is convex, and f is twice differentiable on X . If

- $\nabla f(x^*) = 0$ for some $x^* \in X$, and
- $\mathcal{H}f(x)$ is non-negative definite for all $x \in X$,

then x^* is minimizes $f(x)$ subject to $x \in X$.

¹Send questions and comments to m.tehranchi@statslab.cam.ac.uk. Lecture notes and example sheets can be found at <http://www.statslab.cam.ac.uk/~mike/optimization/>

Notation.

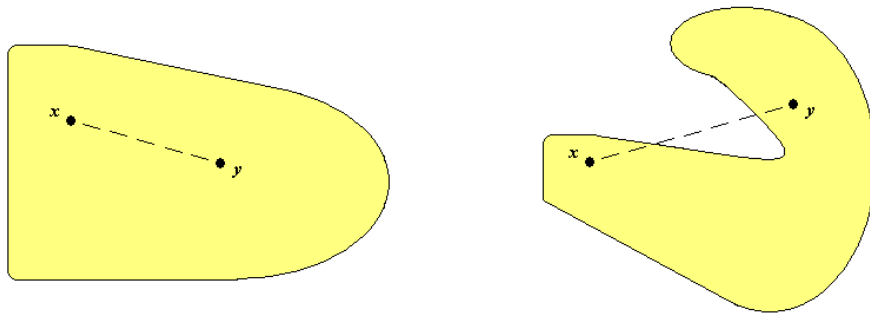
- The gradient: $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)^T$
- The Hessian matrix: $\mathcal{H}f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j}$

An $n \times n$ symmetric matrix A is *non-negative definite* if $v^T A v \geq 0$ for all $v \in \mathbb{R}^n$ or equivalently, if all of the eigenvalues of A are non-negative real numbers.

A subset $X \subseteq \mathbb{R}^n$ is *convex* iff

$$x, y \in X \text{ implies } \theta x + (1 - \theta)y \in X \text{ for all } 0 \leq \theta \leq 1.$$

Below is an example of a convex set on the left, and a set that is not convex on the right.



Example Minimize $x_1^4 + 3x_2^2 - 4x_1$ subject to $x_1, x_2 \in \mathbb{R}$.