IB Optimisation: Lecture 8

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11 May 2020

Last time we used to the simplex algorithm to to

$$P: \text{ maximise } 3x_1 + 2x_2 \quad \text{subject to} \quad \begin{aligned} 2x_1 + x_2 &\leq 4, \quad x_1, x_2 \geq 0 \\ 2x_1 + 3x_2 &\leq 6. \end{aligned}$$

With some work, we computed

initial tableau

	x_1	x_2	z_1 *	z_2 *	
z_1	2	1	1	0	4
z_2	2	3	0	1	6
payoff	3	2	0	0	0

final tableau

	x_1 *	x_2 *	z_1	z_2	
x_1	1	0	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{3}{2}$
x_2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	1
payoff	0	0	$-\frac{5}{4}$	$-\frac{1}{4}$	$-\frac{13}{2}$

Remark. If our linear program is a minimisation (rather than a maximisation), then the test of optimality of step (1) is to stop if the payoff row of the tableau is non-negative (rather than non-positive).

Remark. When the problem is of the form

$$\text{maximise } c^\top x \text{ subject to } Ax \leq b,\ x \geq 0.$$

then the coefficients of the payoff row under the slack variables are minus the corresponding dual variables, where the correspondence is via complementary slackness. And for the final tableau, the feasible dual variables are the Lagrange multipliers for the problem. For the example problem, the Lagrange multipliers are $\lambda_1 = 5/4$ and $\lambda_2 = 1/4$.

Remark. The top two rows of the initial and final tableaux:

$$\begin{pmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 0 & 4 \\ 2 & 3 & 0 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{3}{4} & -\frac{1}{4} & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

Hence, the perturbed problem

$$\text{maximise } 3x_1 + 2x_2 \quad \text{subject to} \quad \begin{array}{rcl} 2x_1 + x_2 & \leq & 4 + \varepsilon_1, \\ 2x_1 + 3x_2 & \leq & 6 + \varepsilon_2 \end{array} \quad x_1, x_2 \geq 0$$

yields the following tableau:

	$*$ x_1	$*$ x_2	z_1	z_2	
x_1	1	0	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{3}{2} + \frac{3}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2$
x_2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$1 - \frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2$
payoff	0	0	$-\frac{5}{4}$	$-\frac{1}{4}$	$-\frac{13}{2} - \frac{5}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2$

.

That means that

$$x_1 = \frac{3}{2} + \frac{3}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2, \quad x_2 = 1 - \frac{1}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2$$

is the optimal solution of the perturbed problem assuming that $x_1 \geq 0$ and $x_2 \geq 0$. The perturbed problem has value $\frac{13}{2} + \frac{5}{4}\varepsilon_1 + \frac{1}{4}\varepsilon_2$, confirming our result that the Lagrange multipliers are shadow prices.

Consider the problem to

$$P: \text{ maximise } c^\top x \text{ subject to } Ax = b, \; x \geq 0.$$

In step 0 of the simplex algorithm, we need to know at least one basic feasible solution. How can we find one efficiently?

Suppose $b \geq 0$, and consider the problem to

$$P' : \text{ minimise } e^\top y \text{ subject to } Ax + y = b, \; x, y \geq 0$$

where $e = (1, \ldots, 1)^\top$.

- Apply the simplex algorithm to P' with initial b.f.s. $(x, y) = (0, b)$.
- If problem P is feasible, then the value of P' is zero, and the simplex algorithm will terminate with final b.f.s $(x, y) = (x_0, 0)$.
- The vector x_0 is a b.f.s. for P!
- This gives rise to the two-phase simplex algorithm.

Example of the two-phase method

Consider the problem to

$$\text{maximise } x_1 - 3x_2 + 5x_3 \text{ subject to} \qquad \begin{aligned} x_1 + x_2 + x_3 &\leq 30 \\ -x_2 + 2x_3 &= 20 \\ -x_1 + 2x_2 + x_3 &\geq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Introduce slack variables z_1 and z_2.

$$\begin{aligned} x_1 + x_2 + x_3 + z_1 \qquad\qquad &= 30 \\ -x_2 + 2x_3 \qquad\qquad &= 20 \\ -x_1 + 2x_2 + x_3 \qquad - z_2 &= 40 \\ x_1, x_2, x_3, z_1, z_2 &\geq 0 \end{aligned}$$

Introduce *artificial* variables y_1 and y_2.

$$\begin{aligned}
x_1 + x_2 + x_3 + z_1 \qquad\qquad\qquad &= 30 \\
-x_2 + 2x_3 \qquad\qquad + y_1 \qquad &= 20 \\
-x_1 + 2x_2 + x_3 \qquad - z_2 \qquad + y_2 &= 40 \\
x_1, x_2, x_3, z_1, z_2, y_1, y_2 &\geq 0
\end{aligned}$$

We solve this new problem with the simplex algorithm. Here's our first tableau.

				*		*	*	
	x_1	x_2	x_3	z_1	z_2	y_1	y_2	
z_1	1	1	1	1	0	0	0	30
y_1	0	-1	2	0	0	1	0	20
y_2	-1	2	1	0	-1	0	1	40
Phase II	1	-3	5	0	0	0	0	0
Phase I	0	0	0	0	0	1	1	0

Notice that we have put both the original objective function (in the Phase II line) and new objective function (in the Phase I line). The point is to manipulate both simultaneously so that when we find an optimal solution to the Phase I problem, we'll already be in shape to use the simplex algorithm on the Phase II problem.

Something is not quite right: the payoff row of Phase I should have zero coefficients for basic variables. Subtract rows 2 and 3 from row 5, yielding the new tableau:

	x_1	x_2	x_3	$\overset{*}{z_1}$	z_2	$\overset{*}{y_1}$	$\overset{*}{y_2}$	
z_1	1	1	1	1	0	0	0	30
y_1	0	-1	2	0	0	1	0	20
y_2	-1	2	1	0	-1	0	1	40
Phase II	1	-3	5	0	0	0	0	0
Phase I	1	-1	-3	0	1	0	0	-60

Okay, now we are ready to apply the simplex algorithm. Since we are trying to minimise, rather than maximise, the Phase I payoff, we look for *negative* coefficients in the payoff row.

The sequence of tableaux are

	x_1	x_2	x_3	$\ast$ z_1	z_2	$\ast$ y_1	$\ast$ y_2	
z_1	1	1	1	1	0	0	0	30
y_1	0	-1	$\boxed{2}$	0	0	1	0	20
y_2	-1	2	1	0	-1	0	1	40
Phase II	1	-3	5	0	0	0	0	0
Phase I	1	-1	-3	0	1	0	0	-60
			$\uparrow$					

	x_1	x_2	x_3 *	z_1 *	z_2	y_1	y_2 *	
z_1	1	$\frac{3}{2}$	0	1	0	$-\frac{1}{2}$	0	20
x_3	0	$-\frac{1}{2}$	1	0	0	$\frac{1}{2}$	0	10
y_2	-1	$\boxed{\frac{5}{2}}$	0	0	-1	$-\frac{1}{2}$	1	30
Phase II	1	$-\frac{1}{2}$	0	0	0	$-\frac{5}{2}$	0	-50
Phase I	1	$-\frac{5}{2}$	0	0	1	$\frac{3}{2}$	0	-30
		↑						

		*	*	*				
	x_1	x_2	x_3	z_1	z_2	y_1	y_2	
z_1	$\frac{8}{5}$	0	0	1	$\frac{3}{5}$	$-\frac{1}{5}$	$-\frac{3}{5}$	2
x_3	$-\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	16
x_2	$-\frac{2}{5}$	1	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	12
Phase II	$\frac{4}{5}$	0	0	0	$-\frac{1}{5}$	$-\frac{13}{5}$	$\frac{1}{5}$	-44
Phase I	0	0	0	0	0	1	1	0

Since the Phase I payoff row has only non-negative entries, we have found an optimal solution

$$(x_1, x_2, x_3, z_1, z_2, y_1, y_2) = (0, 12, 16, 2, 0, 0, 0)$$

to our new problem. And since the artificial variables are zero, we have found a b.f.s of our original problem, namely $(x_1, x_2, x_3, z_1, z_2) = (0, 12, 16, 2, 0)$. (If the Phase I algorithm terminated with an optimal solution in which the artificial variables are strictly positive, the original Phase II problem would have no feasible solution.)

Now we can drop the Phase I row and the columns corresponding to the artificial variables, and proceed as usual.

		*	*	*		
	x_1	x_2	x_3	z_1	z_2	
z_1	$\boxed{\frac{8}{5}}$	0	0	1	$\frac{3}{5}$	2
x_3	$-\frac{1}{5}$	0	1	0	$-\frac{1}{5}$	16
x_2	$-\frac{2}{5}$	1	0	0	$-\frac{2}{5}$	12
Phase II	$\frac{4}{5}$	0	0	0	$-\frac{1}{5}$	-44
	$\uparrow$					

Note that we're maximising now, so we pick our pivot column by choosing a *positive* entry in the payoff row.

	* x_1	* x_2	* x_3	z_1	z_2	
x_1	1	0	0	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{5}{4}$
x_3	0	0	1	$\frac{1}{8}$	$-\frac{1}{8}$	$\frac{65}{4}$
x_2	0	1	0	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{25}{2}$
Phase II	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	-45

and so the value of the original problem is 45 with an optimal solution at $(x_1, x_2, x_3) = (\frac{5}{4}, \frac{25}{2}, \frac{65}{4})$.