IB Optimisation: Lecture 7

Mike Tehranchi

University of Cambridge

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We are still considering the problem to

$$\text{maximise } c^\top x \text{ subject to } Ax = b, \; x \geq 0.$$

Suppose we know one basic feasible solution x_0. Before implementing the simplex algorithm, we need to do some pre-processing of the problem.

- Let $B \subset \{1, \ldots, n\}$ the set of basic indices and N the set of non-basic indices of x_0
- For $x \in \mathbb{R}^n$, use the notation $x = \binom{x_B}{x_N}$ like last time.
- Furthermore, the objective function can be written

$$c^\top x = c^\top x_0 + \mu_{N,0}^\top x_N$$

where $\mu_0 = c - A^\top \lambda_0$ and $\lambda_0 = (A_B^\top)^{-1} c_B$ is the Lagrange multiplier matched to the b.f.s. x_0 by complementary slackness.

- The set of feasible solutions becomes

$$x_B + A_B^{-1} A_N x_N = x_{B,0}, \;\; x_B, x_N \geq 0,$$

Step (0). The initial *simplex tableau* is

$$\Gamma = \begin{array}{|cc|c|} \hline I & A_B^{-1} A_N & x_{B,0} \\ \hline 0 & \mu_{N,0}^\top & -c^\top x_0 \\ \hline \end{array}$$

where I is the $m \times m$ identity.

(1) *Test for optimality.* If $\mu_0 \leq 0$ then STOP! The current b.f.s. is optimal. Otherwise go to step (2).

(2) *Choose the pivot column.* Pick a $j \in N$ such that $\mu_{j,0} > 0$. (Rule of thumb: Pick j such that $\mu_{j,0}$ is largest)

(3) *Choose the pivot row.* Look within the pivot column j and find the $i \in B$ which minimises $x_{i,0}/\Gamma_{i,j}$ over all $i \in B$ such that $\Gamma_{i,j} > 0$. If $\Gamma_{i,j} \leq 0$ for all i, then STOP! the problem is unbounded

(4) *Perform the pivot operation.* Move to the next b.f.s. as follows:

- Replace row i with (old row i)$/\Gamma_{i,j}$.
- Replace row k with (old row k) $-$ (old row i) $\times \Gamma_{k,j}/\Gamma_{i,j}$, for all $k \neq i$

Now return to step (1).

Remarks.

1. For the initial b.f.s we have $x_{i,0} > 0$ and $x_{j,0} = 0$. For the next b.f.s we have $x_{i,1} = 0$ and $x_{j,1} = x_{i,0}/\Gamma_{i,j} > 0$.
2. Indeed, the pivot operation is simply Gaussian elimination, rewriting the problem in terms of the new basis $B_1 = B_0 \cup \{j\} \backslash \{i\}$.
3. After the pivot, the first m rows of the $(n+1)$-th (far-right) column of the tableau is just the basic part of the new b.f.s. x_1. The bottom right entry $\Gamma_{m+1,n+1}$ is now $-c^\top x_1$, i.e. minus the value of the ojective function at the new b.f.s.

Remark 4. Suppose $\Gamma_{i,j} \leq 0$ for all $i \in B$ in step (3). Then picking one $i \in B$ and for $r > 0$ let $x_r = x_0 + r(\delta_j - \Gamma_{i,j}\delta_i)$ where where $\delta_{k,\ell} = 1$ if $k = \ell$ and 0 otherwise.
That is, x_r replaces $x_{i,0}$ with $x_{i,r} = x_{i,0} - r\Gamma_{i,j}$, replaces $x_{j,0} = 0$ with $x_{j,r} = r$, and leaves all other entries unchanged.
Note x_r is feasible since $x_r \geq 0$ and

$$\begin{pmatrix} I & A_B^{-1}A_N \end{pmatrix} x_r = x_0$$

However, $c^\top x_r = c^\top x_0 + r\mu_{j,0} \to \infty$ as $r \to \infty$. In particular, the problem is unbounded.
(This proves the claim from last lecture.)

Example. Consider the linear program to

$$P: \text{ maximise } 3x_1 + 2x_2 \quad \text{subject to} \quad \begin{array}{rcl} 2x_1 + x_2 & \leq & 4, \\ 2x_1 + 3x_2 & \leq & 6. \end{array} \quad x_1, x_2 \geq 0$$

Before using the simplex algorithm, we do some side computations to see what is going on. The dual problem is to

$$D: \text{ minimise } 4\lambda_1 + 6\lambda_2 \quad \text{subject to} \quad \begin{array}{rcl} 2\lambda_1 + 2\lambda_2 & \geq & 3, \\ \lambda_1 + 3\lambda_2 & \geq & 2 \end{array} \quad \lambda_1, \lambda_2 \geq 0$$

We introduce slack variables as usual to both problems, and list all of the basic solutions, paired by complementary slackness:

$$x_1 v_1 = x_2 v_2 = z_1 \lambda_1 = z_2 \lambda_2 = 0.$$

The graph and table shows the set of feasible solutions of both problems.

We see that the optimal solution is at point D where $(x_1, x_2) = (3/2, 1)$ corresponding to the dual solution $(\lambda_1, \lambda_2) = (5/4, 1/4)$. Note that at this point both the primal and dual solutions are feasible.

(0) Start with an initial b.f.s. $(x_1, x_2, z_1, z_2) = (0, 0, 4, 6)$ and put the problem in the *simplex tableau*.

			$*$	$*$	
	x_1	x_2	z_1	z_2	
z_1	2	1	1	0	4
z_2	2	3	0	1	6
payoff	3	2	0	0	0

Notice that we are now at point A.

(1) *Test for optimality.* Not optimal, since the payoff row $(3, 2, 0, 0)$ is not non-positive.

(2) *Choose the pivot column.* Since $3 > 2$, the rule of thumb says let x_1 enter basis.

(3) *Choose the pivot row.* There are two choices. Choosing the first row sends x_1 to $4/2 = 2$, corresponding to point B. If we tried the second row, sending x_2 to $6/2 = 3$, we would go to the infeasible point C.

			*	*	
	x_1	x_2	z_1	z_2	
z_1	$\boxed{2}$	1	1	0	4
z_2	2	3	0	1	6
payoff	3 ↑	2	0	0	0

(4) *Perform the pivot operation.*

	$*$			$*$	
	x_1	x_2	z_1	z_2	
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	2
z_2	0	2	-1	1	2
payoff	0	$\frac{1}{2}$	$-\frac{3}{2}$	0	-6

The new b.f.s is $(x_1, x_2, z_1, z_2) = (2, 0, 0, 2)$, which is point B.

(1) Still not optimal since the payoff row is not non-positive. Equivalently, the point B is not feasible for the dual problem.

(2) The only possibility is to choose the second column about which to pivot. The means that x_2 will enter the basis.

(3) Since $2/(1/2) = 4 > 2/2 = 1$, we pivot about the second row. In the diagram, this means we go to the point D, rather than to the point F which is infeasible for the primal problem.

	$*$			$*$	
	x_1	x_2	z_1	z_2	
x_1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	2
z_2	0	$\boxed{2}$	-1	1	2
payoff	0	$\frac{1}{2}$	$-\frac{3}{2}$	0	-6
		$\uparrow$			

(4) Perform the pivot.

	* x_1	* x_2	z_1	z_2	
x_1	1	0	$\frac{3}{4}$	$-\frac{1}{4}$	$\frac{3}{2}$
x_2	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	1
payoff	0	0	$-\frac{5}{4}$	$-\frac{1}{4}$	$-\frac{13}{2}$

The new b.f.s is $(x_1, x_2, z_1, z_2) = (\frac{3}{2}, 1, 0, 0)$ which is point D.

(1) Our latest b.f.s. is optimal since the payoff row is non-positive. STOP!