IB Optimisation: Lecture 11

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The max flow problem is a linear program. It is natural to consider its dual problem. The Lagrangian is

$$
\begin{aligned}
L(x, z, \lambda, \mu) =& v + \sum_i \lambda_i \left(\sum_j x_{ij} - \sum_j x_{ji} \right) - \lambda_1 v + \lambda_n v \\
& + \sum_{ij} \mu_{ij} (c_{ij} - x_{ij} - z_{ij}) \\
=& \sum_{ij} c_{ij} \mu_{ij} + v(1 - \lambda_1 + \lambda_n) + \sum_{ij} x_{ij} (\lambda_i - \lambda_j - \mu_{ij}) \\
& - \sum_{ij} \mu_{ij} z_{ij}
\end{aligned}
$$

The set of feasible Lagrange multipliers is

$$\Lambda = \{(\lambda, \mu) : \sup_{x,z \geq 0} L(x, z, \lambda, \mu) < \infty\}$$
$$= \{(\lambda, \mu) : \lambda_1 - \lambda_n = 1, \lambda_i - \lambda_j \leq \mu_{ij},\ \mu_{ij} \geq 0\}$$

The dual problem is then

$$\text{minimise } \sum_{ij} c_{ij}\mu_{ij} \text{ subject to } \lambda_1 - \lambda_n = 1, \lambda_i - \lambda_j \leq \mu_{ij},\ \mu_{ij} \geq 0$$

The complementary slackness conditions says that if x is optimal for the primal problem and (λ, μ) is optimal for the dual, then

$$\lambda_i - \lambda_j < \mu_{ij} \Rightarrow x_{ij} = 0 \text{ and } x_{ij} > 0 \Rightarrow \lambda_i - \lambda_j = \mu_{ij}$$

and

$$\mu_{ij} > 0 \Rightarrow x_{ij} = c_{ij} \text{ and } x_{ij} < c_{ij} \Rightarrow \mu_{ij} = 0.$$

The max-flow min-cut theorem says that there is a set S with $1 \in S$ and $n \in \bar{S}$, that is $(S, \bar{S})$ is a cut, such that the optimal dual variables are given by

$$\lambda_i = \left\{ \begin{array}{ll} 1 & \text{if } i \in S \\ 0 & \text{if } i \in \bar{S} \end{array} \right\}$$

and

$$\mu_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } i \in S, j \in \bar{S} \\ 0 & \text{otherwise} \end{array} \right\}.$$

Claim. If the Ford–Fulkerson algorithm terminates, then the final value of the flow is optimal.

- By weak duality, the value of any flow is bounded above by the capacity of any cut.
- The algorithm stops when there are no more augmenting paths, that is, when there does not exist a sequence of nodes $1 = i_0, \ldots, i_r = n$ such that either there is spare capacity $c_{i_k,i_{k+1}} - x_{i_k,i_{k+1}} > 0$ in an arc from the source to the sink of positive flow $x_{i_{k+1},i_k} > 0$ going the wrong way from the sink to the source.
- If there are no augmenting paths, then we can construct a set S such that $(S, \bar{S})$ is a cut of capacity equal to the value v of the flow. This shows that the flow must be maximal.

Does this algorithm eventually terminate?

- Yes, *if* both the initial flow $(x_{ij})_{i,j}$ and the capacities $(c_{ij})_{i,j}$ are rational numbers.
- To see why, multiply both the initial flow and the capacities by the lowest common denominator.
- Then if there exists an augmenting path, then the increment δ additional flow is at least one.
- Hence, at each iteration of the algorithm, the flow value increases my at least one.
- But since the value of the flow is bounded (for instance, by the sum of the capacities) we must eventually run out of augmenting paths and hence have found the optimal flow.

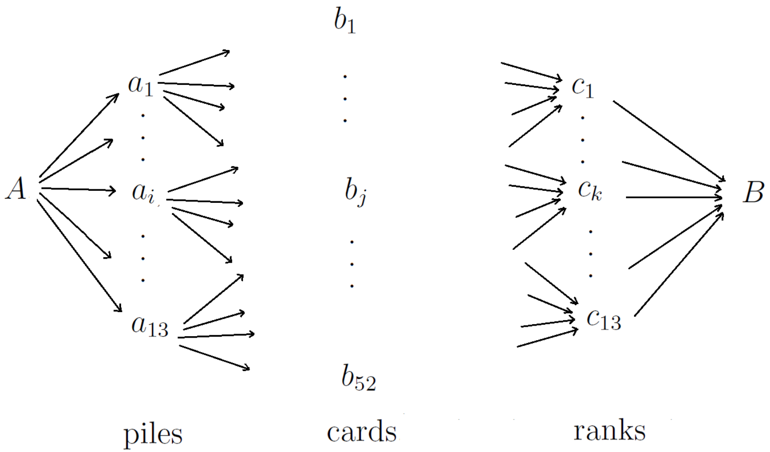
(Non-examinable) If either the initial flow or the capacities is irrational, then the algorithm may not terminate.

Example.

Suppose that a standard deck of 52 playing cards is dealt into 13 piles of 4 cards each. Show that it is possible to select exactly one card from each pile such that the selected cards contain one card of each of the 13 ranks Ace, 2, . . . , 10, Jack, Queen, and King.

Solution. Build a network with nodes A, $a_1, \ldots, a_{13}$, $b_1, \ldots, b_{52}, c_1, \ldots, c_{13}, B$ as follows. The nodes $(a_i)_i$ represent the 13 piles, the nodes $(b_j)_j$ represent the 52 cards, and the nodes $(c_k)_k$ represent the 13 ranks. The node A is the source and node B the sink.

- Put an arc of capacity 1 from the source A to each of the nodes a_i.
- Put an arc of infinite capacity from a_i to b_j if card j is in stack i. There will be exactly four arcs leaving of each node a_i and exactly one arc entering each node b_j.
- Put an arc of infinite capacity from b_j to c_k if card j has rank k. There will be exactly one arc leaving each of the nodes b_j and exactly four arcs entering each of the nodes c_k.
- Put an arc of capacity 1 from each node c_k to the sink B.


b_1
a_1
A
a_i
b_j
c_1
c_k
B
a_{13}
c_{13}
b_{52}
piles
cards
ranks

Claim: the minimum capacity of any cut 13.

Proof. Remove each arcs of the form (A, a_i) for $i = 1, \ldots, 13$, i.e. set $S = A$. The capacity of this cut is 13. We now show that every other cut has a capacity of at least 13.

- Consider a cut of finite capacity. Since the capacity is finite, it will not contain any arc of the form (a_i, b_j) or (b_j, c_k). Suppose that there are r arc of the form (c_k, B) in the cut. That means that there are $s = 13 - r$ nodes c_k are still connected to the source A.
- Each of these c_k is connected to four nodes b_j, for a total of $4s$ nodes b_j still connected to the source A.
- Each group of four nodes b_j is connected to at least one node a_i, so there are a total of *at least* $(4s)/4 = s$ nodes a_i still connected to the source A.
- To complete the cut, we need to remove at least s arcs of the form (A, a_i). The capacity of the cut then is at least $r + s = 13$.

By the max-flow min-cut theorem, there exists a flow of value 13. Now, since the capacities are either 1 or infinity, the Ford-Fulkerson starting with initial flow 0 will terminate. Hence there is an optimal flow in which the flow along every arc is an integer. And by the capacity constraint, the only possibility is 0 or 1.

Let x be the optimal flow thus constructed. For each i, the optimal flow from the source A to a_i is $x_{A,a_i} = 1$. By conservation of flow, there is a j such $x_{a_i,b_j} = 1$. And by conservation of flow, for such j there exists k such that $x_{b_j,c_k} = 1$. In this way, we match every pile a_j with exactly one rank c_k.