Stochastic Calculus Michael Tehranchi
Lecture 5 - Lent 2011

Now we will discuss the strong Markov property of Brownian motion. But first need to recall some preliminary notions. In what follows, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space.

Definition. A *filtration* $(\mathcal{F}_t)_{t\geq 0}$ is an increasing family of sigma-fields, i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $0 \leq s < t$.

Definition. A *stopping time* is a random variable $T : \Omega \to \mathbb{R}_+ \cup \{+\infty\}$ such that $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Definition. A process $(X_t)_{t\geq 0}$ is *adapted* if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

Definition. For a stopping time T, we use the notation

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}$$

If you are new to these ideas, you should check the following:

- If there is a constant $t_0 \geq 0$ such that $T(\omega) = t_0$ for all ω, then T is stopping time and $\mathcal{F}_T = \mathcal{F}_{t_0}$.
- For any stopping time T, the collection of sets $\mathcal{F}_T$ is a sigma-field on Ω.
- If S and T are stopping times, so is $S \wedge T$.
- If S is a stopping time and $T = S$ a.s., then T is also a stopping time under the additional assumption that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.
- The left continuous process $(\mathbb{1}_{\{t\leq T\}})_{t\geq 0}$ is adapted if and only if T is a stopping time.

Now, if X is adapted and T is a stopping time, it is natural to consider the function X_T. On the event $\{T < \infty\}$, this function is defined by $X_T(\omega) = X_{T(\omega)}(\omega)$. And on the event $\{T = \infty\} \cap \{(X_n)_n \text{ converges }\}$, it is defined by $X_T(\omega) = \lim_n X_n(\omega)$. Otherwise, we set $X_T(\omega)$ so some arbitrary value, since in practice we are not interested in this case. Now, is X_T an $\mathcal{F}_T$-measurable random variable? The answer is always yes in discrete time, but in continuous time, we need an additional measurability assumption:

Definition. A process X is called *progressively measurable* if for all $t \geq 0$, the map $(s, \omega) \mapsto X_s(\omega)$, which send $[0, t] \times \Omega$ into $\mathbb{R}$, is $\mathcal{B}[0, t] \otimes \mathcal{F}_t$-measurable.

Here is a very general theorem about progressive measurability, whose proof is omitted.

Theorem. If X is adapted and if $(t, \omega) \to X_t(\omega)$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-measurable, then X has a progressively measurable modification.

The above result is more general than we need for this course. Here is a less general result, which we will find very useful.

Proposition. If X is adapted and if X has right-continuous (or left-continuous) sample paths, then X is progressively measurable.

Proof. Suppose X is adapted and right-continuous. Fix $t > 0$, and let $\tau_n : [0, t] \to [0, t]$ be defined by $\tau_n(t) = t$ and

$$\tau_n(s) = kt/n \text{ when } (k-1)t/n \leq s < kt/n \text{ for some } 1 \leq k \leq n$$

Note that $\tau_n(s) \downarrow s$ for each $s \in [0,t]$. Now consider $X^n_s = X_{\tau_n(s)}$. Since X is right-continuous $X_s(\omega) = \lim_n X^n_s(\omega)$ for all (s,ω). So it enough to show $(s,\omega) \mapsto X^n_s(\omega)$ is measurable. But

$$X^n_s(\omega) = \sum_{k=1}^{n} \mathbb{1}_{[(k-1)t/n, kt/n)}(s) X_{kt/n}(\omega) + \mathbb{1}_{\{t\}}(s) X_t$$

Clearly, the function $\mathbb{1}_{[(k-1)t/n,kt/n)}$ is $\mathcal{B}[0,t]$ measurable. Also, the random variable $X_{kt/n}$ is $\mathcal{F}_t$-measurable, since $kt/n \le t$ and since X is assumed to be adapted. So the process X^n is $\mathcal{B}[0,t] \otimes \mathcal{F}_t$ measurable for each n as promised. □

Proposition. If T is a stopping time and X is progressively measurable, then $(X_{t\wedge T})_{t\ge 0}$ is progressively measurable. Furthermore, X_T is a $\mathcal{F}_T$-measurable random variable.

Proof. For each $t \ge 0$, the map $(s,\omega) \mapsto (s \wedge T(\omega), \omega)$ sending $[0,t] \times \Omega$ into itself is $\mathcal{B}[0,t] \otimes \mathcal{F}_t$ measurable, since T is a stopping time. Then $(s,\omega) X_{s\wedge T}(\omega) = X_{s \wedge T(\omega)}(\omega)$ is a composition of measurable maps, and hence measurable.

Finally, let B be a Borel set, and note the identity

$$\{X_T \in B\} \cap \{T \le t\} = \{X_{t\wedge T} \in B\} \cap \{T \le t\}.$$

Since $X_{t\wedge T}$ is $\mathcal{F}_t$-measurable, $\{X_{t\wedge T} \in B\}$ is in $\mathcal{F}_t$. Also, $\{T \le t\}$ is in $\mathcal{F}_t$ since T is a stopping time. This shows $\{X_T \in B\}$ is in $\mathcal{F}_T$. □

Definition. A Markov process $X = (X_t)_{t\ge 0}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\ge 0})$ is a process such that

$$\mathbb{E}[f(X_{t+u})|\mathcal{F}_t] = \mathbb{E}[f(X_{t+u})|X_t]$$

for all $t, u \ge 0$ and bounded measurable f.

Definition. A strong Markov process X is such that

$$\mathbb{E}[f(X_{T+u})|\mathcal{F}_T] = \mathbb{E}[f(X_{T+u})|X_T]$$

for all $u \ge 0$, finite stopping times T, and bounded measurable f.

For discrete time/discrete state space processes, the Markov property implies the strong Markov property. This is no longer the case for continuous time/ continuous space processes, as an example sheet problem demonstrates. However, Brownian motion does have the strong Markov property. We will see this next lecture.