Stochastic Calculus Michael Tehranchi
Example sheet 4 - Lent 2015

Problem 1. Construct a filtered probability space on which a Brownian motion W and an adapted process X are defined and such that

$$dX_t = \frac{X_t}{t}dt + dW_t, \quad X_0 = 0.$$

Is X adapted to filtration generated by W? Is W a Brownian motion in the filtration generated by X? [Hint: See example sheet 1, problem 3.]

Problem 2. (a) Prove Bihari's inequality: Suppose f is satisfies the inequality

$$f(t) \le a + \int_0^t k(f(s))ds \text{ for all } t \ge 0$$

for a constant a and where the function k is positive and increasing and $k \circ f$ is locally integrable. Show that

$$\int_a^{f(t)} \frac{du}{k(u)} \le t \text{ for all } t \ge 0.$$

(b) Show that if

$$0 \le f(t) \le \int_0^t k(f(s))ds \text{ for all } t \ge 0$$

and

$$\int_0^r \frac{du}{k(u)} = \infty \text{ for all } r > 0$$

then $f(t) = 0$ for all $t \ge 0$.
(c) Prove Osgood's uniqueness theorem: Suppose for every $n \ge 1$ there exists a positive, increasing function k_n such that

$$\|b(x) - b(y)\| \le k_n(\|x - y\|) \text{ for all } x, y \in \mathbb{R}^d, \|x\| \le n, \|y\| \le n.$$

Furthermore, suppose that for all n,

$$\int_0^r \frac{du}{k_n(u)} = \infty \text{ for all } r > 0.$$

Show that for each $x_0 \in \mathbb{R}^d$ there is at most one solution to the ODE

$$\dot{x} = b(x)$$

with $x(0) = x_0$.
(d) Formulate and prove a similar pathwise uniqueness theorem for the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t.$$

Problem 3. Let b be bounded and measurable. Use Girsanov's theorem to construct a weak solution to the SDE

$$dX_t = b(X_t)dt + dW_t$$

over the finite (non-random) time interval $[0, T]$.

Problem 4. By inspecting the proof of Novikov's condition, show that if there exists an increasing sequence $t_n \to \infty$ such that

$$\mathbb{E}(e^{\frac{1}{2}(\langle M\rangle_{t_n} - \langle M\rangle_{t_{n-1}})}) < \infty$$

then $\mathcal{E}(M)$ is a true martingale. Hence, show that if $b : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and satisfies the linear growth condition

$$\|b(x)\| \leq C(1 + \|x\|) \text{ for all } x \in \mathbb{R}^d$$

for a constant $C > 0$, then the SDE

$$dX_t = b(X_t)dt + dW_t$$

has a weak solution over any finite time interval $[0, T]$. You may want to use this form of Jensen's inequality:

$$e^{\int_s^t a_u \, du} \leq \frac{1}{t-s} \int_s^t e^{(t-s)a_u} du.$$

Problem 5. Show that the SDE

$$dX_t = 3X_t^{1/3}dt + 3X_t^{2/3}dW_t, \quad X_0 = 0$$

has strong existence, but not pathwise uniqueness.

Problem 6. Let X be the Markov process associated with the scalar SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t.$$

Let the C^2 function $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ satisfy the PDE

$$\frac{\partial u}{\partial t} = b(x)\frac{\partial u}{\partial x} + \frac{\sigma(x)^2}{2}\frac{\partial^2 u}{\partial x^2} + g(x)$$

with boundary condition $u(0, x) = 0$ for all x. Assuming that g is bounded, and that u is bounded on any strip $[0, t] \times \mathbb{R}$, then show that

$$u(t, x) = \mathbb{E}\left[\int_0^t g(X_s)ds | X_0 = x\right].$$

Problem 7. Find the unique strong solution of the SDE

$$dX_t = \frac{1}{2}X_t dt + \sqrt{1 + X_t^2}dW_t, \quad X_0 = x.$$

[Hint: consider the change of variables $Y_t = \sinh^{-1} X_t$.]

Problem 8. Let W and B be independent Brownian motions, and let

$$X_t = e^{-W_t}\left(x + \int_0^t e^{W_s}dB_s\right)$$

Show that there exists a Brownian motion Z such that

$$dX_t = \frac{1}{2}X_t dt + \sqrt{1 + X_t^2}dZ_t, \quad X_0 = x.$$

Use the previous problem to find the density function of the random variable $\int_0^t e^{W_s}dB_s$.

Problem 9. Let X be a solution of the SDE

$$dX_t = X_t g(X_t) dW_t$$

where g is bounded and non-random $X_0 > 0$.
(a) Show that $\mathbb{P}(X_t > 0 \text{ for all } t \geq 0) = 1$. [Hint: Apply Itô's formula to $\mathcal{E}(\int g(X)dW)^{-1}X$.]
(b) Show that $\mathbb{E}(X_t) = X_0$ for all $t \geq 0$.
(c) Fix a non-random time horizon $T > 0$. Show that there exists an equivalent measure $\hat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ and a $\hat{\mathbb{P}}$-Brownian motion $\hat{W}$ such that

$$dY_t = Y_t g(1/Y_t) d\hat{W}_t$$

where $Y_t = 1/X_t$.

Problem 10. (square-root diffusion) Let W be an n-dimensional Brownian motion, and define an n-dimensional process X to be the solution to the SDE

$$dX_t = -X_t dt + dW_t$$

with $X_0 = x \in \mathbb{R}^n$. If $R_t = \|X_t\|^2$, show that there exists a scalar Brownian motion Z such that

$$dR_t = (n - 2R_t)dt + 2\sqrt{R_t} dZ_t.$$

Problem 11. By finding the stationary solution of the Fokker–Plank PDE, find a formula for the invariant density of the scalar SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

assuming it exists. Apply your formula to the process R in the previous question. Comment of your answer in light of example sheet 3, problem 9.

Problem 12. Consider the SDE

$$dX_t = X_t^2 dW_t.$$

(a) Use example sheet 3, problem 5 to construct a weak solution.
(b) Verify that both $u^1(t,x) = x\left[2\Phi\left(\frac{1}{x\sqrt{t}}\right) - 1\right]$ and $u^2(t,x) = x$ solve the PDE

$$\frac{\partial u}{\partial t} = \frac{x^4}{2}\frac{\partial u^2}{\partial x^2}, \quad u(0,x) = x$$

(c) Which of these solutions corresponds to $u(t,x) = \mathbb{E}(X_t | X_0 = x)$?

Problem 13. (a) Suppose X is a weak solution of the SDE $dX_t = b(X_t)dt + \sigma(X_t)dW_t$. Show that the process

$$f(X_t) - \int_0^t \left[b(X_s)f'(X_s) + \frac{1}{2}\sigma(X_s)^2 f''(X_s)\right] ds$$

is a local martingale for all $f \in C^2$.
(b) Let X be a scalar, continuous, adapted process such that

$$f(X_t) - \int_0^t \left[b(X_s)f'(X_s) + \frac{1}{2}\sigma(X_s)^2 f''(X_s)\right] ds$$

is a local martingale for each $f \in C^2$. Suppose σ is continuous and $\sigma(x) > 0$ for all x. Show that there exists a Brownian motion such that $dX_t = b(X_t)dt + \sigma(X_t)dW_t$. [Hint: Consider example sheet 3 problem 3.]

Problem 14. (Brownian bridge) Let W be a standard Brownian motion.
(a) Let $B_t = W_t - tW_1$. This is called a Brownian bridge. Can you see why? Show that $(B_t)_{t\in[0,1]}$ is a continuous, mean-zero Gaussian process. What is the covariance $c(s,t) = \mathbb{E}(B_s B_t)$?
(b) Is B adapted to the filtration generated by W?
(c) Let

$$dX_t = -\frac{X_t}{1-t}dt + dW_t, \quad X_0 = 0.$$

Verify that $X_t = (1-t)\int_0^t \frac{dW_s}{1-s}$ for $0 \le t < 1$. Show $X_t \to 0$ as $t \uparrow 1$. [Hint: show that there exists a Brownian motion Z such that $\int_0^t \frac{dW_s}{1-s} = Z_{t/(1-t)}$ and apply the Brownian strong law of large numbers.]
(d) Show that X is a continuous, mean-zero, Gaussian process with the same covariance as B, i.e. X is a Brownian bridge.